2012

A permutational triadic approach to jazz harmony and the chord/scale relationship

John Bishop
Louisiana State University and Agricultural and Mechanical College

A PERMUTATIONAL TRIADIC APPROACH TO JAZZ HARMONY AND THE CHORD/SCALE RELATIONSHIP

A Dissertation
Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in

The School of Music

by

John Bishop
B.M., Berklee College, 1990
M.M., University of Louisville, 2004
December 2012

To Quentin Sharpenstein

The harmonic, simple, and direct triad is the true and unitrisonic root of all the most perfect and most complete harmonies that can exist in the world. It is the root of even thousands and millions of sounds….The triad is the image of that great mystery, the divine and solely adorable Unitrinity (I cannot think of a semblance more lucid). All the more, therefore, should theologians and philosophers direct their attention to it, since at present they know fundamentally little, and in the past they knew practically nothing about it….It is much employed in practice and, as will soon be seen, stands as the greatest, sweetest, and clearest compendium of musical composition….This triad I have observed since boyhood (with only God and nature as my guides), I now study it by way of a pastime, and I hope to see it perfected with God's help, to Whom be praise forever.
− Johannes Lippius, *Synopsis of New Music (Synopsis Musicae Novae)*.

God has wrought many things out of oppression. He has endowed his creatures with the capacity to create—and from this capacity has flowed the sweet songs of sorrow and joy that have allowed man to cope with his environment and many different situations.
Jazz speaks for life. The Blues tell a story of life's difficulties, and if you think for a moment, you will realize that they take the hardest realities of life and put them into music, only to come out with some new hope or sense of triumph.
This is triumphant music.
Modern jazz has continued in this tradition, singing songs of a more complicated urban experience. When life itself offers no order of meaning, the musician creates an order and meaning from the sounds of the earth, which flow through his instrument.
Much of the power of our Freedom Movement in the United States has come from this music. It has strengthened us with its sweet rhythms when courage began to fail. It has calmed us with its rich harmonies when spirits were down.
− Dr. Martin Luther King, Jr., Opening Address to the 1964 Berlin Jazz Festival.

For music…we could envisage the question of how to 'perform' abstract algebraic structures. This is a deep question, since making music is intimately related to the expression of thoughts. So we would like to be able to express algebraic insights, revealed by the use of K-nets or symmetry groups, for example, in terms of musical gestures. To put it more strikingly: 'Is it possible to play the music of thoughts?
− Guerino Mazzola and Moreno Andreatta, "Diagrams, Gestures and Formulae in Music."

ACKNOWLEDGEMENTS

I would like to express my appreciation for my family and their support and inspiration throughout this process. My wife, Diễm Mai Bishop and my children Sarah Ngọc Vân Bishop and Jonah Ngọc Qúy Bishop; my parents, John Bishop Jr., Nancy Bishop, Vũ Ngô Dũng, and Vũ Hà Lan Nha; and my siblings, Karen Bishop-Holst and Vũ Ngô Hà Ngọc. Without their help, this work would not be possible. I hope they see it as an expression of my love for them.

I would also like to thank those at Louisiana State University. My advisor, Dr. Robert Peck for inspiring me to pursue studies in mathematics and for his patience while I was learning. Also, Dr. Jeffrey Perry and Dr. David Smyth were instrumental in my understanding of Schenkerian techniques and who were fundamental in my development. Dr. Willis Deloney, Dr. William Grimes and Dr. Brian Shaw, members of the jazz studies department provided the utmost support for my studies. I took great pride in working with them.

The concept of using triads as an improvisational tool was first introduced to me my Jon Damian, Chan Johnson, and Larry Sinibaldi nearly three decades ago; this is the genesis of my musical problem addressed here. Luthier Abe Rivera changed the course of my life by reinstating music as my primary focus. I also thank Ann Marie de Zeeuw, my for supported my interest in music theory, and Dale Garner whose influence instilled in me a great respect for mathematics and the desire to continue to learn.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF EXAMPLES

LIST OF DEFINITIONS

LIST OF ANALYSES

SYMBOLS

Musical Notation

Δ	Major
$-$	Minor
$\longrightarrow$	Dominant-action V^{7}/X resolution
$\dashrightarrow$	Dominant-action ${}^{sub}V^{7}/X$ resolution
└──┘	II—V
└ - - ┘	Chromatic II—V
( )	deceptive resolution (dominant-action chords)
└──┴──┘ X^{T}	Tonic System
M.I.	Modal Interchange
[ ]	Relative II^{-} related only its dominant, not to the global key

Mathematical Notation

$:=$	Defines
$\boldsymbol{X}$	Set
$\lvert\boldsymbol{X}\rvert$	Absolute value (size, or order) of the set $\boldsymbol{X}$
$\subset$	Subset
$\emptyset$	Null (or empty) set
$\cup$	Set union
$\cap$	Set intersection
$\mathbb{Z}$	The set of integers
$\mathbb{Z}_n$	The set of integers modulo n
X	Group
$<g>$	Group generator
$\mid$	Such that
$\lvert X\rvert$	Absolute value (size, or order) of the group X
A_n	Alternating group on n elements
C_n	Cyclic group of order n
D_n	Dihedral group of order n
O	Octahedral rotational symmetry group
Oh	Octahedral full symmetry group
S_n	Full symmetric group on n elements
Th	Tetrahedral group
T/I	Transposition and inversion group
V_4	Klein 4-group
X_k^j	j copies of the group X acting on k elements
$X_k^{[j]}$	The specific j copy of X acting on k elements

$\cong$	Isomorphic to
$<$	Subgroup
$\lhd$	Normal subgroup
$\times$	Direct product
$\rtimes$	Left normal factor semidirect product
$\Leftrightarrow$	Commutative
$X^{\#}$	The set of non-identity group members

ABSTRACT

This study provides an original triadic theory that combines existing jazz theory, in particular the chord/scale relationship, and mathematical permutation group theory to analyze repertoire, act as a pedagogical tool, and provide a system to create new music. Permutations are defined as group actions on sets, and the sets used here are the constituent consonant triads derived from certain scales. Group structures provide a model by which to understand the relationships held between the triadic set elements as defined by the generating functions. The findings are both descriptive and prescriptive, as triadic permutations offer new insights into existing repertoire. Further, the results serve as an organizational tool for the improviser and composer/arranger. In addition to the ability to describe individual triadic musical events as group actions, we also consider relationships held among the musical events by considering subgroups, conjugacy classes, direct products and semidirect products. As an interdisciplinary study, it is hoped that this work helps to increase the discourse between those in the music subdisciplines of mathematical music theory and jazz studies.

.

CHAPTER 1. INTRODUCTION, PRELIMINARIES, AND HISTORICAL CONTEXT

1.1. Introduction

This study posits an original triadic theory using mathematical permutation group theory, wherein group actions over a set of triads derived from the chord/scale relationship inform a systematic organizational method applicable to the creation of an improvisational scheme, analytical technique, or composition method. A mathematical group, simply defined, is a function acting on a set of elements. The resulting mapping of the set onto itself is called an action, which induces a permutation of the set itself.

As an interdisciplinary study, this work aims to bridge the divide between jazz research and recent mathematic-music-theoretical work. The intended readership is that of jazz theorists as well as musicians. As such, mathematical concepts and definitions are introduced throughout the document to provide the reader with a mathematical example coupled with a discussion of a well-known jazz theoretical concept whenever possible. The section on set determination acts as a primer into jazz theory for readers with limited exposure to those concepts.

The branch of mathematics included herein is that of applied mathematics: not mathematics for mathematics sake—there are no new mathematical concepts presented in this study—its novelty lies in the practical application of existing mathematical concepts to an art form. John Rahn provides appropriate language regarding the scope of mathematical applications in a musical context:

> This essay is written by a music theorist who is not a mathematician. It aims, at least, to be comprehensible to both music theorists and mathematicians, and hopes

> to be useful to both. It [his essay, and in this case, this document] contains some new music theory, but no new mathematics.[1]

In the bebop era (ca.1941-55), musicians began to employ triads as a device to organize harmonic tensions {9, 11, 13} and their alterations. As the jazz genre progressed, new triadic practices developed. Harmonies with root motions based on equal divisions of the octave came into vogue during the post-bop period (beginning ca. 1959), a technique commonly attributed to John Coltrane and the players influenced by his work.[2] Eventually, triads, emancipated from their functional harmonic underpinnings, came to act as stand-alone musical entities. This appears in three guises: (1) as superimpositions (reharmonizations) over a composition's stated harmony, acting as an instrument for melodic improvisation or chordal accompaniment, as in Coltrane's use of chromatic mediant harmonies over the harmonic framework of a standard tune. (2) As an agent to affect modal interchange (modal mixture) from collections other than the diatonic. For example, E^{Δ}/C serves as a modal interchange replacement for $IV^{\Delta 7}$ in the key of G major, where the replacement chord derives from the C Lydian augmented scale (third mode of the real melodic minor[3] built on the pitch A) thus coloring the $C^{\Delta 7}$ as a $C^{\Delta 7\sharp 5}$. (3) As a compositional force wherein the inclusion of triadic structures marks important events within the context of a composition's overall harmonic scheme, as in Coltrane's "Giant Steps." Since the early 1970s, the use of triads-over-bass-notes has been widely accepted as a compositional and improvisational device. The bass note may be a member of the triad, act as a chordal seventh, or be foreign to the triad. Compositions using the technique exist where the triad-over-bass-note

---

[1] John Rahn, "Cool Tools: Polysemic and Non-Commutative Nets, Subchain Decompositions and Cross-Projecting Pre-Orders, Object-Graphs, Chain-Hom-Sets and Chain-Label-Hom-Sets, Forgetful Functors, free Categories of a Net, and Ghosts." *Journal of Mathematics and Music* 1, no.1 (2007): 7.

[2] The post-bop period emerged during the mid-1960's and includes such recordings as Miles Davis's *Miles Smiles*, McCoy Tyner's *The Real McCoy*, Wayne Shorter's releases as a leader and the music of Joe Henderson. Many post-bop musicians played an instrumental role in the development of the jazz fusion style in the 1970's.

[3] The real melodic minor, referred to colloquially as "jazz minor," is the ascending traditional melodic minor. In the real melodic minor, no adjustments are made to the descending form.

structures are seemingly unrelated to each other.[4] The opening to Mick Goodrick's composition "Mevlevia" is an example, in which the first five bars contain the sonorities B$^{\Delta}$/E—C$^{\Delta}$/E♭—D♭$^{\Delta}$/D—D$^{\Delta}$/C♯—E♭$^{\Delta}$/C. Nguyên Lê's "Isoar" is another example, where the first five bars unfold over B^{-7}—B♭$^{\Delta 7}$—F$^{\Delta}$/A—B♭$^{\Delta}$/G♭—A$^{\Delta}$/C—D^{-7}. We shall see this in subsequent analyses of Mick Goodrick's reharmonization of "I've Got Rhythm," and Kenny Wheeler's composition "Ma Belle Hélène," where triads-over-bass-notes are the exclusive source of harmonic vocabulary.

The chord/scale relationship is the means by which to reconcile two musical domains, melody and harmony. Jazz students first learn that musical domains are the horizontal (melody) and vertical (harmony). When they are told, "to solo over the D^{-} expanse in Miles Davis's 'So What,' play D Dorian," they relate the harmony with a scale; musical domains, however, are rarely so neatly separable. We can generate chords from scales, as in basic diatonic theory; conversely, we may generate scales from chords. An example of the latter is the chord G$^{7\ 9,\sharp 11,13}$. Harmonic notation, assuming ascending generic thirds, indicates the pitch collection {G,B,D,F,A,C♯,E}—a modal presentation of an ascending D melodic minor scale.

Harmonic function affects chord/scale determination. The manner in which harmony and melody interact owes much to musical context within a global setting. For example, the chord C^{7} may contain the unaltered tensions (9,13), as is the case when C^{7} functions as V^{7}, invoking Mixolydian as the chord/scale relation. However, C^{7} can also function as a secondary dominant, for example, V^{7}/VI^{-} in the key of A♭ major. Here, harmonic function generates the altered tensions (♭9,♭13), and the chord/scale relationship must be modified to fit this new image of C^{7}.

[4] Nontraditional triadic approaches occur in other modern genres as well. Rebeca Mauleón-Santana describes the practice in Latin piano montunos. See, Rebeca Mauleón-Santana, *101 Montunos.* (Petaluma, CA: Sher Music Company, 1999), 128-9. Kevin Bond and Kevin Powell feature triadic techniques as stylistic hallmarks in contemporary gospel music. In fact, during the author's decades of performing contemporary gospel music, I observed the triad-over-bass-note technique to be the preferred method by which gospel musicians communicate harmonic structures. For example, C^{-7} is described as E♭$^{\Delta}$/C and G$^{7\ \flat 9,13}$ is described as E$^{\Delta}$/G where members of E$^{\Delta}$ relate to G^{7}: E = 13, G♯ = A♭ = ♭9, B = 3.

Therefore, we cannot narrowly define the chord/scale relationship as strictly a melodic or harmonic device.

Return to the students playing "So What," and say one student plays a chordal instrument and that he/she is accompanying a soloist. As for the soloist, the music created is based on the relationship between the composition's stated harmony and its corresponding scale. Now consider the chordal player. If however, the chordal player were to provide harmonic support with non-tertian chord voicings, e.g. a stack of generic fourths, and run these voicings through the prism of D Dorian, is the resultant music domain based on a chord/chord relationship, a scale/scale relationship, or does it retain the chord/scale relationship? It is the latter. Although the accompanist is producing simultaneities based on the pitches contained in D Dorian over D^{-} harmony, the chord/scale relationship remains the definitive factor of musical domain.

We define a set in its mathematical sense, as the consonant triads generated by a specific scale, upon which the group will act, henceforth referred to as the scale's *constituent consonant triads*. This practice folds we tie mathematical concepts into an existing musical theoretical framework that is easily understood by jazz musicians. Having defined a set, we then apply a function to the set to form a mathematical group. Permutations generated by the group can be used to identify, analyze, and create musical events based on relationships within the group's structure. Regarding the concepts of function verses relation, Bert Mendelson, in *Introduction to Topology*, describes their difference as follows:

> A function may be viewed as a special case of what is called a relation. We are accustomed to thinking of one object being in a given relation to another; for example, Jeanne is the sister of Sam or silk purses are more expensive than sow's ears. To say that the number 2 is less than the number 3, or $2 < 3$, is thus to say that (2,3) is one of the number pairs (x,y) for which the relation "less than" is true.[5]

[5] Bert Mendelson, *Introduction to Topology* , 3rd ed. (New York: Dover Publications, Inc.), 15.

In music theory, we also find function to be a special case of a given musical relation. Take two pitches, C_4 and G_4. We say that G_4 is an upper fifth to C_4 and call this a *relation*. To generate G_4 from C_4, we invoke a transposition up seven semitones, T_7, and call this a *function*. Similarly, we can generate G_3 from C_4 through motion by directed interval class (i.c.) −5, and call this either a *relation* or a *function*. Nevertheless, can we not say that T_7 is both a *function* and a *relation*? Or describe the musical instruction "in bar 8 move to the upper fifth of C" as a *function*, as it generates the next event, and accept the answer "an upper fifth" to the question "what is the chord in bar 8" as a *relation*?

To expand this concept, we say that in the key of C major, the *relation* of F^{-} to C^{Δ} is that of a modally inflected subdominant, *generated* by the binary function on C^{Δ}, transposition by i.c. 5, and reversing chord quality (parity).[6] Additionally, we may *generate* F^{-} from C^{Δ} through inversion (I_0) in pitch-space, $(C, E, G) \xrightarrow{I_0} (C, A\flat, F)$ *and* say F^{-} and C^{Δ} *relate* by I_0. How do we define the difference between *function* and *relation*? Perhaps a better question is not how but why we differentiate between the two. In the present study, the group G is generated by the function(s), $<f>$, acting on a set of triads $\boldsymbol{S}$, written $G := (\boldsymbol{S}, f)$, which *generates* a set of k

[6] The incorporation here of the term parity agrees with usage in triadic-based neo-Riemannian theories from which many of the ideas in this study derive. The basic definition of parity in mathematics is the term that states if an object is even or odd. Note the general use of the term object. In mathematics, an object can be an integer, where parity describes whether the integer is even or odd. An object can be a permutation, where parity describes the number of transpositions held in permutation decomposition. (We shall see more on this use during the discussion of alternating groups.) An object can be a function, where parity describes how its values change when its arguments are exchanged with their negations. An object can be coordinates in Euclidean space with dimensions ≥ 2, and so on. Julian Hook, in "Uniform Triadic Transformations," *Journal of Music Theory* 46, nos. 1-2 (2002): 57-126, provides the following notation for triadic transformations: $<\sigma, x, y>$. The possible values for x and y are {0…11} to denote levels of transposition where x = major triads and y = minor triads. The variable σ shows if the transformation preserves the triad's mode (shown with the symbol +), or if the transformation reverses the triad's mode (shown with the symbol −). Therefore, parity describes whether a constituent triad is major or minor and what happens to the quality under some operation.

permutations on the elements of ***S***; we also consider the triads contained in ***S*** to be *k-related*, and name this relation according to the degrees of *f* that generate *k*.

This dissertation comprises five chapters. The introductory chapter, Chapter 1, includes a literature review, a section on mathematical preliminaries, and a discussion of nontraditional triadic usage within a historical context. Chapter 2 addresses set definition and contains an overview of functional harmony as it applies to jazz, followed by a section on chord/scale relationships. A survey of existing triad-based improvisational methods closes Chapter 2. Chapter 3 addresses various group actions on triadic sets. At the beginning of Chapter 3, a scale roster presents nine unique scales that generate the triadic sets required to investigate the group actions. Groups are delineated by the size (cardinality) of the sets upon which they act; sets of orders 3 through 8 are discussed and modeled on geometric objects that includes two-dimensional *n*-gons and certain Platonic solids in three-dimensions. Chapters 2 and 3 contain analyses of existing compositions using the Permutational Triadic Approach; material in Chapter 4 focuses on application by offering examples of reharmonization and improvisational schemes. Chapter 4 closes with discourse on a specific harmonic system often found in the music of John Coltrane. These triadic permutations tie into the neo-Riemannian transformations geometrically modeled on Cayley digraphs and toroidal polygons. Chapter 5 serves as a conclusion and offers questions for further application/research.

1.2. Literature Review

The literature review is organized into subsections that cover the components of the Triadic Permutational Approach separately. The subsection on jazz literature includes topics such as functional harmony, the chord/scale relationship, and triadic specific improvisational methods. The triadic- theory literature subsection covers neo-Riemannian transformational theory, work pertaining to triadic chromaticism, and studies on voice-leading. The mathematical literature subsection covers topics limited to group-theoretical applications and generalized transformational theories. The literature review addresses the sources pertinent to the central argument presented within this study and does not claim to be an exhaustive account of the existing literature on any of the topics. For example, there are a number of reliable sources pertaining to the chord/scale relationship, and the inclusion of a particular method in this study does not infer that the author implicitly endorses that method over any other. Technical terms introduced in this section are defined in subsequent sections.

1.2.1. Jazz Literature

Barrie Graff and Richard Nettles provide a theory of functional harmony, analytical techniques, and a method for chord/scale determination.[7] The functional-harmony analytical symbols and the chords/scale theory used in this work come from Graff/Nettles.[8] Wayne Naus describes non-functional harmony as an extension of the functional harmony theories found in Graff/Nettles.[9] Ron Miller's study of modal harmony includes modes from scales other than the diatonic, for example, what Miller calls "altered diatonic scales": real melodic minor, harmonic minor, harmonic major, and real melodic minor ♯5. Miller's study provides us with the ability to (1) include altered diatonic scales as modal generators, and (2) allow altered diatonic modal harmonies to act as modal interchange chords (modal mixture).[10]

---

[7] Barrie Nettles and Richard Graf, *The Chord Scale Theory and Jazz Harmony* (Rottenberg: Advance Music, 1997). Functional harmony refers to harmony discernible in a key. This includes diatonic harmony, applied dominants, modal interchange, and tonic systems (chromatic mediant related harmonies) as long as the tonic system has a functional harmony as its object of resolution.

[8] Chord/scale relationship is a generic term for the relationship between a chord and an associated scale while chord/scale theory is the specific name given to the work of Graff and Nettles.

[9] Wayne Naus, *Beyond Functional Harmony* (n.p.: Advance Music, 1998).

[10] Ron Miller, *Modal Jazz Composition and Harmony, vol.1* (Rottenburg: Advance Music, 1996).

1.2.2. Chord/Scale Relationship Literature

Chord/scale relationships derive from a number of musical criteria. Jamey Aebersold's "scale syllabus," which he attributes to David Baker, is a method of chord/scale determination based solely on chord quality.[11] Aebersold lists common chord qualities, some with harmonic tensions, then recommends a number of possible scales. While the pedagogical importance of the "scale syllabus" is undeniable, it neglects functional harmonic considerations. Therefore, this study adheres to the Graff/Nettles model, wherein a chord's function, as well as its quality, informs chord/scale determinations.

In *The Lydian Chromatic Concept of Tonal Organization for Improvisation*, George Russell describes a collection of seven scales, organized around a Lydian-based "parent scale," thereby defining a set of possible scale choices over a given harmony.[12] He provides a systematic method by which to organize scales based on their varying degrees of dissonance between the harmony and the scale. With Russell's method, we also gain the ability to transition from one scale type, henceforth referred to as a *scale genre* (diatonic, altered diatonic, symmetric), to another scale genre based upon an initial scale choice (the parent scale).

1.2.2. Triadic Specific Methods for Jazz Improvisation

The following sources are improvisational methods based on the manipulation of triads. There are two reasons for this focus: first, the triad is uniquely aurally identifiable; second, a player attempting to perform the repertoire presented here should already possess an understanding of triads on their instrument. Gary Campbell and Walt Weiskopf, in separate

---

[11] Jamey Aebersold, *A New Approach to Jazz Improvisation* (New Albany, IN: Aebersold Jazz Inc., 1975) and *Jazz Handbook* (New Albany, IN: Aebersold Jazz Inc., 2010).

[12] George Russell, *The Lydian Chromatic Concept of Tonal Organization for Improvisation* (New York: Concept Publishing Company, 1959).

publications, describe an improvisational technique that uses *triad pairs* as the generators of melodic lines. Chord/scale relationships dictate from which scales the triad pairs derive. For example, if G^7 is the stated harmony, G Mixolydian is one possible corresponding scale, of which triads D^- and E^- are subsets; therefore, the set $\{D^-, E^-\}$ is a viable triad pair for use over G^7.[13]

George Garzone's *Triadic Chromatic Approach* uses an arbitrary triad choice, where the selection of triads is free of any chord/scale prerequisite. In Garzone's approach, he allows any triad to follow any other triad given that no two consecutive triads appear in an invariant inversional position: $\frac{5}{3}$ cannot follow $\frac{5}{3}$; $\frac{6}{3}$ cannot follow $\frac{6}{3}$; $\frac{6}{4}$ cannot follow $\frac{6}{4}$.[14] Suzanna Sifter's recent work on upper-structure triads focuses on triads superimposed over seventh-chords as a method to manage harmonic tensions (extensions) in piano voicings.[15] While not a study dedicated entirely to triads, David Liebman's *A Chromatic Approach to Jazz Harmony and Melody* presents a method by which the musician may increase the amount of chromaticism in one's playing by using polytonal triads to generate melodic lines and harmonic structures.[16]

1.2.3. Triadic Theory

Neo-Riemannian theory plays an important role in our current understanding of triadic theory, and more specifically, of consonant triads in chromatic settings. It also offers transformational analytical tools beyond the traditional transposition and inversion operators.[17]

---

13 Gary Campbell, *Triad Pairs for Jazz: Practice and Application for the Jazz Improviser* (n.p.: Alfred Publishing, n.d.); Walt Weiskopf, *Intervallic Improvisation, the Modern Sound: A Step Beyond Linear Improvisation* (New Albany, IN: Aebersold Jazz Inc., 1995).
14 George Garzone, *The Music of George Garzone and the Triadic Chromatic Approach*, DVD (n.p.: Jody Jazz, n.d.).
15 Suzanna Sifter, *Using Upper-Structure Triads* (Boston: Berklee Press, 2011).
16 David Liebman, *A Chromatic Approach to Jazz Harmony and Melody* (Rottenburg: Advance Music, 2001).
17 Although the use of Riemannian transformations in jazz literature is a relatively recent development, use of Riemannian concepts in European jazz scholarship existed as early as 1953. See Alfred Baresel, *Jazz-Harmonielehre* (Trossingen: M. Hohner, 1953). See also, Renate Imig, *Systeme der Funktionsbezeichnung in den*

The amount of literature of neo-Riemannian theory is extensive.[18] However, the work on jazz-based neo-Riemannian theory and mathematically-based triadic theories is less extensive. Representative examples of such research includes Guy Capuzzo's investigation of the intersection between guitarist Pat Martino's concept of guitar fretboard organization and neo-Riemannian theory,[19] and Keith Waters's neo-Riemannian based analysis of Miles Davis's composition "Vonetta," where he takes neo-Riemannian theories and allies the to seventh-chord structures derived from the real melodic minor collection. Waters also employs neo-Riemannian analytical techniques in two-dimensional and three-dimensional geometric spaces to model the [0148] hexatonic subset he finds important in jazz repertoire after ca. 1960.[20] Maristella Feustle also takes up the use of neo-Riemannian techniques for the analysis of seventh-chords in post-bop jazz.[21]

Richard Cohn and Jack Douthett investigate the relationships of triads derived from symmetric scales, and use mathematical concepts and geometric objects to model those relationships. Cohn's related work is rooted in the Weitzmannian[22] and Riemannian traditions and focuses on the triad's role in chromaticism in the music of the middle- to late-nineteenth century. Cohn provides numerous geometric examples, one being his Hyper-Hexatonic System,

---

Harmonielehren seit Hugo Riemann (Düsseldorf: Gesellschaft zur Förderung der systematischen Musikwissenschaft, 1970).

[18] See, for example, Brian Hyer, "Reimag(in)ing Riemann," *Journal of Music Theory* 39, no.1 (1995): 101–13, and Journal of Music Theory's special issue on neo-Riemannian theory, *Journal of Music Theory* 42, no.2 (1998); Edward Gollin and Alexander Rehding, eds., *The Oxford Handbook of Neo-Riemannian Music Theories* (Oxford: Oxford University Press, 2011).

[19] Guy Capuzzo, "Neo-Riemannian Theory and the Analysis of Pop-Rock Music," *Music Theory Spectrum* 26, no.2 (2004): 177-200; "Pat Martino's *The Nature of the Guitar*: An Intersection of Jazz Theory and Neo- Riemannian Theory." *Music Theory Online* 12, no.1 (February, 2006). http://mtosmt.org/issues/mto.06.12.1capuzzo.pdf (accessed March 1, 2010).

[20] Keith Waters, "Modeling Diatonic, Acoustic, Hexatonic, and Octatonic Harmonies and Progressions in 2- and 3-Dimensional Pitch Spaces; or Jazz Harmony after 1960,"*Music Theory Online* 16, no.3 (May, 2010). http://mto.societyofmusictheoty.org/mto.10.16.3.waters.williams.html (accessed June 2, 2010)

[21] Maristella Feustle, "Neo-Riemannian Theory and Post-bop Jazz: Applications of an Extended Analytical Framework for Seventh Chords" (master's thesis, Bowling Green University, 2005).

[22] Carl Friedrich Weitzmann, *Der übermässige Dreiklang* (Berlin: T. Trautweinschen, 1853).

which draws upon Weitzmann's work on the augmented triad. In this system, the four unique augmented triads act as a source set, which partitions all twenty-four consonant triads into four "Weitzmann regions" [Cohn's term] through parsimonious voice leading techniques.[23] Two source augmented triads that flank each hexatonic region, when taken as a set union, obtain a unique hexatonic collection.[24] Figure 1 is an adaptation of Cohn's Hyper-Hexatonic System, substituting the harmonic notation for consonant triads used in the remainder of this document for Cohn's labels.

Possible applications to jazz are readily apparent. The major triads from Coltrane's "Giant Steps" live in the "Western" region, and the triads in the bridge to "Have You Met Miss. Jones" live in the "Southern" region. The Hyper-Hexatonic System explains the definition of hexatonic regions as a set based on their constituent consonant triads. The scale collection generated by the set union of a region's triads represents a possible chord/scale choice applicable to the harmonies contained within that region. Therefore, we gain the ability to unfold triadic permutations from the "Western" region over the changes to "Giant Steps."

---

[23] Richard Cohn, *Audacious Euphony: Chromaticism and the Consonant Triad's Second Nature* (Oxford: Oxford University Press, 2012): 59-81

[24] Richard Cohn, "Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions," *Music Analysis* 15, no.1 (1996): 9-40. See also, "Weitzmann's Regions, My Cycles, and Douthett's Dancing Cubes," Music Theory Spectrum 22, no.1 (2000): 89-103 [94-6]. See also, "Square Dances with Cubes," *Journal of Music Theory* 42, no.2 (1998): 238-96; "Neo-Riemannian Operations, Parsimonious Trichords, and their "*Tonnetz*" Representations," *Journal of Music Theory* 41, no.1 (1997): 1-66.

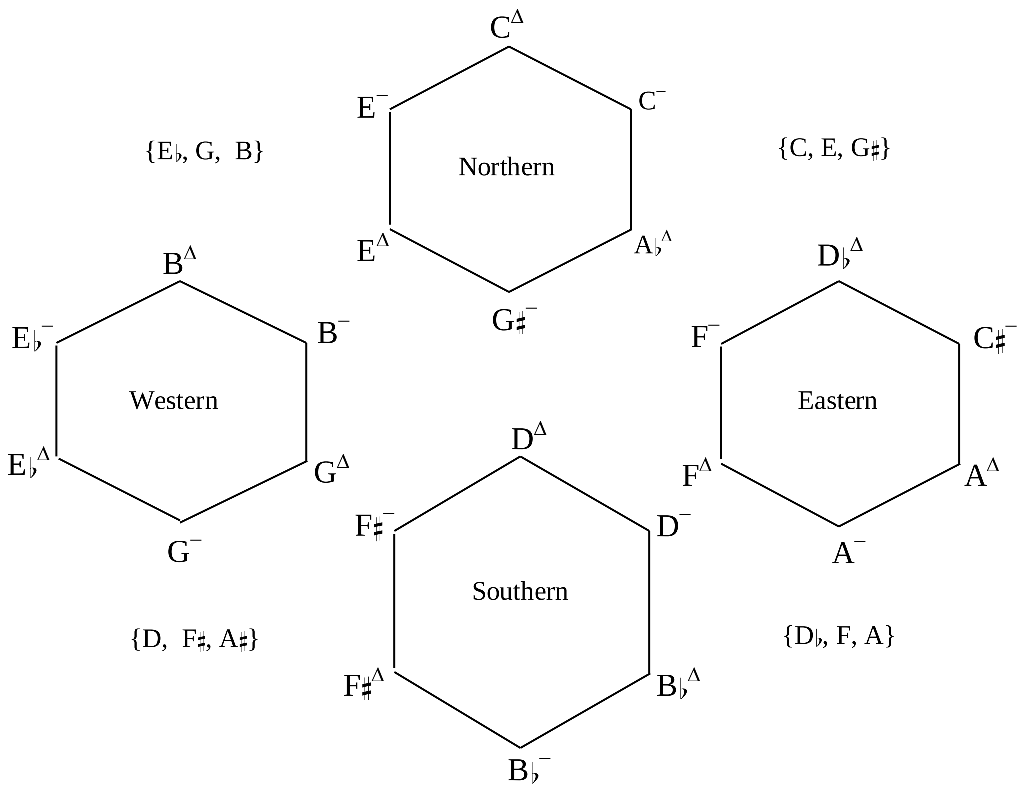


Figure 1. Cohn's Hyper-Hexatonic System[25]

Douthett employs a graph-theoretical approach to construct mode graphs showing parsimonious triads associated with the hexatonic, octatonic and enneatonic collections. He then tiles a torus with a discrete lattice, calling it his "Chicken-Wire Torus" in reference to the resulting hexagonal faces. The Chicken-Wire Torus is the geometric dual of the *Tonnetz*. Douthett describes the symmetries of the torus using group-theoretical concepts.[26] He illustrates octahedral symmetry as symmetries of the octahedron, cube, and a composite figure displaying their geometric duality.[27]

---

[25] Cohn, "Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions," 24.
[26] Jack Douthett and Peter Steinbach, "Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition," *Journal of Music Theory* 42, no.2 (1998): 241-63.

[27] Jack Douthett, "Filtered Point-Symmetry and Dynamical Voice-Leading," in *Music Theory and Mathematics: Chords, Collections and Transformations*, edited by Jack Douthett, Martha M. Hyde, and Charles J. Smith, 72-106

Dmitri Tymoczko also includes octahedral geometric duality in his voice-leading theories, offering an intensive study of geometric models using elaborate networks and lattices (in the form of chord lattices and a scale lattices) to model musical transformations.[28] Some lattices take the form of concatenated cubes in higher spatial dimensions where he describes networks defined by the cubes' edges and vertices.[29] In *A Geometry of Music*, Tymoczko devotes an entire chapter to jazz; he unfortunately suspends the rigorous mathematical descriptions in this section.

1.2.4. Group Theory Literature

An overview of pertinent group-theoretical literature follows. Iannis Xenakis, in *Formalized Music*, explains the rotational symmetry group of a cube, which he uses as an organizational system in his composition *Nomos Alpha*.[30] Robert Peck explains the rotational symmetries of the cube in group-theoretical terms in his analysis of *Nomos Alpha*.[31] Peck also provides additional work addressing many other groups and includes discussions on their actions over symmetric scale collections.[32] Alissa Crans, Thomas Fiore and Ramon Satyendra describe the musical actions of two dihedral groups—the transposition and inversion group and the

(Rochester: University of Rochester Press, 2008). The figure displaying the octahedron's and the cube's geometric duality is shown in this paper's section on octahedral symmetry.

[28] Dmitri Tymoczko, "The Generalized Tonnetz," *Journal of Music Theory* 56, no.1 (2012): 6-10.
[29] Dmitri Tymoczko, "Scale Networks and Debussy," *Journal of Music Theory* 48, no.2 (2004): 219-294; *A Geometry of Music* (Oxford: Oxford University Press, 2011).
[30] Iannis Xenakis, *Formalized Music* (Stuyvesant, NY.: Pendagron, 1992): 219-21.
[31] Robert Peck, "Toward an Interpretation of Xenakis's *Nomos Alpha*"
[32] See, Robert Peck, "Imaginary Transformations," *Journal of Mathematics and Music* 4, no. 3 (2010): 157-71; "Generalized Commuting Groups," *Journal of Music Theory* 54, no.2 (2010): 143-77.

neo-Riemannian group—where they show that both are isomorphic[33] to the dihedral group of order 24.[34] Paul Zweifel generalizes scales using group theory making extensive use of cyclic groups.[35]

1.3. Mathematical Preliminaries

The following preliminaries address the Permutation Triadic Approach's mathematical components. This section contains definitions, a preliminary example of group actions on a familiar musical object, and an overview of the neo-Riemannian group.

Definition 1. Set

A set defines an inclusionary relationship of elements within a single definitive criterion.[36] We may define a set for any type of object, for example, $\mathbb{Z}$ is the set of all integers and $\mathbb{R}$ is the set of all real numbers, both of which are examples infinite sets. Certain musical sets, such as the diatonic collection under octave equivalence, or triads that derive from specific scale collections are examples of finite sets. The salient relationship between sets in present study is the determination of set-element membership. Define the set $\boldsymbol{N} := \{0\ldots11\}$, the set of integers modulo 12 or $\mathbb{Z}_{12}$, which is mapped to the pitch classes in chromatic-pitch-class space, the set $\boldsymbol{P} := \{C,C\sharp,D,\ldots B\}$. Define the set $\boldsymbol{C}$ as pitch classes of the (0♯, 0♭) diatonic,

[33] A definition of isomorphism is provided in the mathematical preliminaries section of this study.
[34] Alissa S. Crans, Thomas M. Fiore, and Ramon Satyendra, "Musical Actions of Dihedral Groups," *The American Mathematical Monthly* 116, no.6 (2009): 479-495.
[35] Paul F. Zweifel, "Generalized Diatonic and Pentatonic Scales: A Group-Theoretic Approach," *Perspectives of New Music* 34, no.1 (1996): 140-161.
[36] This discussion covers well-known set theoretical concepts applicable to the present study. For sources dealing with set theory in musical contexts see, Allen Forte, *The Structure of Atonal Music* (New Haven: Yale University Press, 1973); John Rahn, *Basic Atonal Theory* (Englewood Cliffs, NJ: Prentice-Hall, 1981); Joseph N. Straus, *Introduction to Post-Tonal Theory*, 3rd ed. (Englewood Cliffs, NJ: Prentice-Hall, 2004).

corresponding to the elements in ***N***, ***C*** := {0,2,4,5,7,9,11}. Set ***C*** is described in purely musical terminology as the C major scale. We are now able to answer the question of element membership. Is pitch class (p.c.) 4 a member of ***C*** (written p.c. 4 ∈ ***C***)? The answer is yes. However, p.c. 6 is not a member of ***C*** and we write p.c. 6 ∉ ***C***. Other pertinent set relationships include the following: set union, shown with the symbol ∪; set intersection, shown with the symbol ∩; set cardinality, written as |***X***|; superset, ⊃; and subset ⊂. Define the set ***K*** as the triad pair ***K*** := {C^{Δ}, D^{-}}, and the set ***L*** := {E^{-},F^{Δ}}. ***K*** ∪ ***L*** = {C^{Δ}, D^{-}, E^{-}, F^{Δ}}. Then, specifically in regards to p.c. content, is ***K*** ∪ ***L*** ∈ ***C***? The answer is true; moreover, ***C*** ⊃ ***K*** and ***C*** ⊃ ***L*** (which reads as follows: ***C*** is a superset of ***K*** and ***C*** is a superset of ***L***); ***K*** ⊂ ***C*** and ***L*** ⊂ ***C*** (***K*** is a subset of ***C*** and ***L*** is a subset of ***C***). Define the set ***M*** := {0,1,3,5,6,8,10), the p.c. content of a 5♭ diatonic, the D♭ major scale. Therefore, ***C*** ∩ ***M***, the intersection of sets ***C*** and ***M***, equals {0,5}, which consists of pitch classes 0 (p.c. C) and 5 (p.c. F). Set cardinality is the set's size, the number of elements contained within the set; therefore, |***C***| = 7 and |***K***| = 2.

Definition 2. Group

A group, for this definition labeled Φ, is an order pair Φ := (***Q***, *f*) where ***Q*** represents a set and *f* is a function (action) on ***Q***. A group must hold all of the following properties:

(1) The set must be closed under *f*.
(2) *f* is associative.
(3) There is an identity element, shown as *i*.
(4) Each element *x* has an inverse, shown as x^{-1}.
We shall now redefine a group, using set-theoretical language; numeration of set-theoretical definitions corresponds to those in the group-theoretical definition.
(1) For all $x, y \in \Phi, x \cdot y \in \Phi$.[37]
(2) For all $x, y, z \in \Phi, (x \cdot y) \cdot z = x \cdot (y \cdot z)$.
(3) There exists $i \in \Phi$ such that, for all $x \in \Phi, x \cdot i = i \cdot x = x$.
(4) For all $x \in \Phi$, there exists $x^{-1} \in \Phi$ such that $x \cdot x^{-1} = x^{-1} \cdot x = i$.[38]

[37] The symbol · shows a multiplicative action on group members. *x* · *y* reads as, do *x* to Φ then do *y* to Φ.
[38] Adapted from F.J. Budden, *The Fascination of Groups* (Cambridge: Cambridge University Press, 1972), 73-4.

Group-actions describe bijective (one-to-one and onto) symmetries of a set.[39] For instance, the so-called *symmetric group of the set* consists of all mappings of the set onto itself. This group action type, especially on a finite set, is considered a permutation group. The notation that shows group actions on a set, and, if required, the specific resulting permutation(s), takes the form,

$$(\boldsymbol{Q}_{(n)}, G_{(x)}): y.$$

Assuming our set represents a scale or some other referential pitch collection, $\boldsymbol{Q}_{(n)}$ is the set, where n represents the set's pitch-level in pitch-class space. G is the group that is acting on $\boldsymbol{Q}_{(\mathrm{n})}$, where ($x$) represents the *order* (size) of G. *Order* describes the number of unique permutations within a group. y denotes the permutation(s) of ($\boldsymbol{Q}_{(n)}$, $G_{(x)}$) and y is called a member of ($\boldsymbol{Q}_{(n)}$, G_x). Musically, groups appear in numerous ways.

1. Group-actions as concatenations:

$$((\boldsymbol{Q}_{(n)}, G_x): y), ((\boldsymbol{Q}_{(n)}, G_x): z)$$

Here, one permutation of ($\boldsymbol{Q}_{(n)}$, G_x) follows another (z follows y). However, neither the set nor the group is required to be the same for each iteration. For instance, a diatonically generated set may precede an octatonic set, or a set generated by $\mathrm{Oct}_{(0,1)}$ may precede a set generated by $\mathrm{Oct}_{(2,3)}$. Similarly, the group acting on the set in the first iteration may differ from the group acting on the same set in the second iteration.

2. Group actions over a single stated harmony (k):

$$\frac{\boldsymbol{Q}_{(n)},\ G_{(x)}: y}{k}$$

[39] A bijection is a function that provides an exact pairing of the elements of two sets. Every element of one set is paired with exactly one element of the other set; every element of the other set is paired with exactly one element of the first set. There are no unpaired elements and no two elements of one set map to a single element of the other set. In mathematical terms, a bijective function f: $\boldsymbol{X} \rightarrow \boldsymbol{Y}$ is a one to one and onto mapping of a set $\boldsymbol{X}$ to set $\boldsymbol{Y}$. A permutation is a bijective mapping a set to itself. For more on bijective functions, see Moore, 13-4.

Let k be G^7 in the key of C major and put $\boldsymbol{Q}$ as the constituent consonant triads in G Mixolydian and define a group with an action on $\boldsymbol{Q}$ to be played (musically) over k.

3. Group actions over a harmonic progression:

$$\frac{\boldsymbol{Q}_{(n)},\ G_{(x)}: y}{(h, j, k)}$$

Elements (h, j, k) represent harmonies contained in a chord progression, defined for this example as (D^{-7}, G^7, $C^{\Delta 7}$), II^{-7}—V^7—$I^{\Delta 7}$ in the key of C major. The set $\boldsymbol{C}$ was previously defined as a (0♯, 0♭) diatonic. Define the set $\boldsymbol{C}'$ as the set of consonant triads contained in $\boldsymbol{C}$,

$\boldsymbol{C}' := \{C^{\Delta}, D^{-}, E^{-}, F^{\Delta}, G^{\Delta}, A^{-}\}$, and have a group act on $\boldsymbol{C}'$ to create music over (h, j, k).

Group action and permutation are similar concepts, but there is a subtle distinction. Let $\boldsymbol{S}$ be a set and G be a group. In simplest terms, the action of a group element is a member of a homomorphic mapping of $\boldsymbol{S} \times G \rightarrow \boldsymbol{S}$. In contrast, a permutation is merely a rearrangement of the elements of $\boldsymbol{S}$, a mapping of $\boldsymbol{S}$ onto itself. The action of G on $\boldsymbol{S}$ induces a permutation of $\boldsymbol{S}$'s elements, but it is not a permutation, per se. It is rather a matter of perspective. Consequently, both terms are used throughout this document, with sensitivity to that perspective. (This practice appears with frequency in the relevant literature.)

Definition 3. S_n

The full symmetric group on a set of degree n defines all possible permutations on n elements. The order of S_n is $|n!|$.[40] S_n has connotations on our use of geometric shapes that model n elements. $|S_4| = 4! = 1 \times 2 \times 3 \times 4 = 24$. If we were to attempt to model S_4 on a square, we would be required to wound distort, twist, or topologically modify the square to show certain permutations. If we frame our argument as one that only considers continuous rigid motions of

---

[40] Joseph A. Gallian, *Contemporary Abstract Algebra*, 5th ed. (New York: Houghton Mifflin, 2002), 95-6.

the geometric object i.e., not allowing a twist or other distortions, we must consider subgroups of S_4.[41]

Definition 4. Subgroup

If a subset H of group G is itself a group under the operation of G, H is a subgroup of G, written $H < G$. Define the group G as the transposition group T_n as it acts on the set $\boldsymbol{S}$:= {the twelve major triads}. Then define the group H as the function T_3 acting on the set $\boldsymbol{Q}$:= $\{C^{\Delta}, E\flat^{\Delta}, G\flat^{\Delta}, A^{\Delta}\}$.[42] H is a subset of G and H is indeed a group, therefore, $H < G$. However, as Gallian states, "$\mathbb{Z}_n$ under modulo n is not a subgroup of $\mathbb{Z}$ under addition, since addition modulo n is not the operation of $\mathbb{Z}$."[43]

Lagrange's theorem states that if G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. With Lagrange's theorem, we can extrapolate a list of subgroup candidates based on the orders of the subgroups. For instance, if $|G| = 12$, it may potentially have subgroups of order 12, 6, 4, 3, 2, 1.[44]

Definition 5. Cyclic notation

Cyclic notation is the notation used to show permutation representations, consisting of set elements and parentheses, where integers represent set elements, and the parentheses represent orbits of set elements or the stabilization of set elements (i.e., an element is *stabilized* if it does not permute to another member of the set). By convention, cyclic notation omits stabilized elements. If a set element does not appear in the cyclic notation, that element (or elements) is (are) assumed to be stabilized.

---

[41] For more on S_n, see, John T. Moore, *Elements of Abstract Algebra,* 2nd ed. (New York: Macmillan Company, 1967), 77.
[42] The musical transposition and inversion group (T/I) is the group generated by T_n, the mapping of a pitch-object by n semitone(s) and inversion, the mapping of a pitch-object to its $\mathbb{Z}_{12}$ inverse.
[43] Gallian, 59-60.
[44] Gallian., 137-8.

Let us look at examples of cyclic notation using permutations of a set containing three elements {1, 2, 3}. The *identity* group element, *i*, reads in cyclic notation as (1)(2)(3) where each set element maps to itself, equivalent to doing nothing to the set, the set appears as it did prior to applying the function. Cyclic notation showing the permutation (123) reads as follows: 1 maps to the position previously held by 2, 2 maps to the position previously held by 3, and 3 maps to the position previously held by 1 (wrapping around from the last parenthesis to the first). The same permutation could be written (231), or (312), as the action on the set is the same, $1 \mapsto 2$; $2 \mapsto 3$; $3 \mapsto 1$. Permutation (23) reads as follows: 2 maps to the position previously held by 3, 3 maps to the position previously held by 2, and 1 is stabilized; 1 remains at its point of origination (i.e., is stabilized).

Each permutation representation of a group action shows a unique mapping of set elements, generated by some product of the group's generator(s). We give multiple actions as an ordered *n*-tuple. For example, the ordered duple <*x*,*y*> means "do *x*, then do *y*." We will use left functional orthography, in which *x* is followed by *y*.[45]

Definition 6. Orbits and stabilizers

John Dixon and Brian Mortimer provide the following definition,

> When a group *G* acts on a set ***S***, a typical point α is moved by elements of *G* to various other points. The set of these images is called the *orbit* of α under *G*, and we denote it by
> $$\alpha^G := \{\alpha^x \mid x \in G\}.$$
> A dual role is played by the set of elements in G, which fix a specified point α. This is the stabilizer of α in G and is denoted.
> $$G_\alpha := \{x \in G \mid \alpha^x = \alpha\}.$$ [46]

[45] Right functional orthography (where the example would read *y* followed by *x*) occurs in many mathematical texts. This study employs left functional orthography exclusively.

[46] John D. Dixon and Brian Mortimer, *Permutation Groups* (New York: Springer, 1996), 7-9.

The orbit's periodicity, or order, is of great interest, as understanding the correlation between orbit order and the generative permutation aids in the identification of applicable subgroups and the identification of equivalency classes.

Definition 7. Homomorphism

A homomorphism from a group G to a group G' is a mapping δ from G to G' that preserves the group operation; that is, $\delta(ab) = \delta(a)\delta(b)$ for all a,b in G.[47] A homomorphism is a generalized isomorphism (see the following section).

Definition 8. Isomorphism

Isomorphism is a bijective (one-to-one and onto) homomorphism, where the mapping μ from group G to group G', $G \xrightarrow{\mu} G'$, preserves the group operation, $\mu(xy) = \mu(x)\mu(y)$ for all x,y in G.[48] If there exists an isomorphism from G to G' we say G and G' are isomorphic and write $G \cong G'$. For example, the group of symmetries of the set $\boldsymbol{N} := (0\ldots11)$ ($\mathbb{Z}_{12}$) under addition is isomorphic to the set of symmetries of the set $\boldsymbol{P}$ of pitch classes in chromatic space under transposition. Therefore, we write $\boldsymbol{N} \cong \boldsymbol{P}$.

Definition 9. Automorphism

An automorphism is an isomorphism where a mathematical object is mapped to itself. The automorphism group of the set is also referred to as the full symmetric group on the set, written $S_{(x)}$, and has the order $|\mathrm{Sym}(x)| = x!$. As an introductory example, let us consider a rudimentary group structure to which we shall return, the symmetries of an equilateral triangle. For musical relevance, let the three unique octatonic collections be the elements of the set

[47] Gallian, 194-5.
[48] Budden, 118-20.

B := {1 = $\mathrm{Oct}_{(0,1)}$ = {0,1,3,4,6,7,9,10}; 2 = $\mathrm{Oct}_{(1,2)}$ = {1,2,4,5,7,8,10,11}; 3 = $\mathrm{Oct}_{(2,3)}$ = {2,3,5,6,8,9,11,0}. Plot the elements of ***B*** on the triangle's vertices. Listed below each triangle is a permutation representation, i.e., (i, r, r^{-1}, $f_{(a\ldots c)}$) with the corresponding mapping shown in cyclic notation.

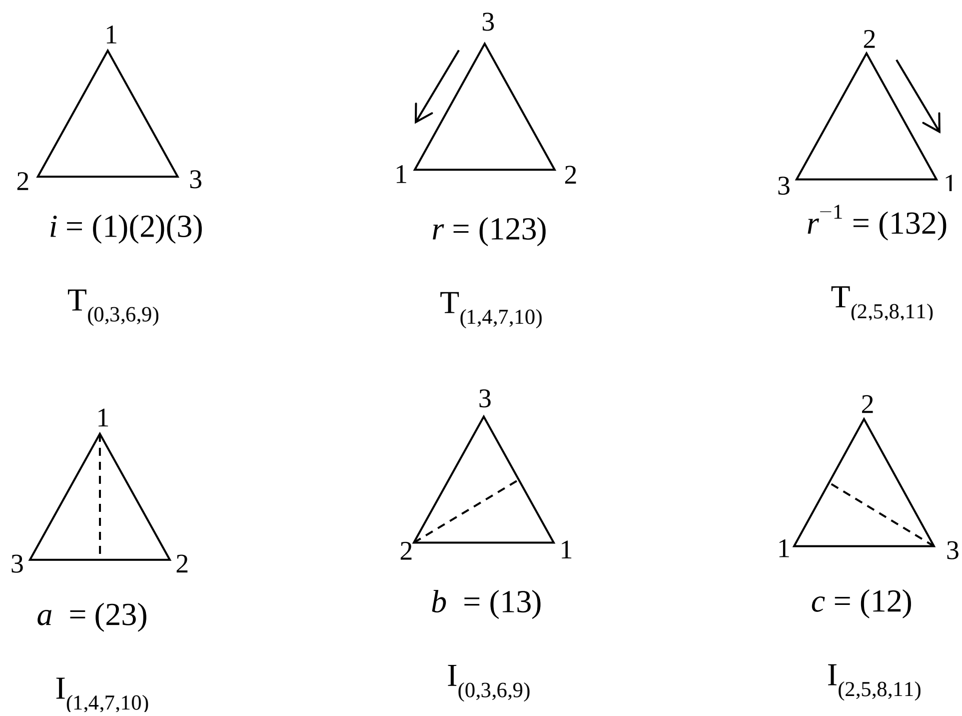


Figure 2. Symmetries of the triangle

There are two types of generators for the full symmetry group of a triangle: rotation and reflection. The group is of order 6, with the following group members: identity (i), rotation by 120° (r), inverse rotation (r^{-1}), and three distinct reflections (f) through axes ($a\ldots c$). The subgroup generated by rotations on n elements is isomorphic to the cyclic group on n elements, written C_n. In the present case, $r^3 = i$, which means r, performed three times returns the identity. Define the group J: = (***B***, C_3), shown in the top row of Figure 2. Adding a generative reflection on ***B*** to J produces the group K, which is isomorphic to the dihedral group of order 6. This group is written D_6, and is shown in the bottom row of Figure 2.[49] Define the group M as the musical transposition and inversion group (*T/I* group) acting on $\boldsymbol{B}$. We have now defined an isomorphism that describes familiar musical operations as a group, $M \cong D_6$.

Definition 10. The Neo-Riemannian Group

Neo-Riemannian theory frequently incorporates thee distinct operations, Parallel (*P*), Relative (*R*), and Leading-tone exchange (*Leitonwechel*) (*L*).

$$\text{Parallel } (P) = \left(x^{\Delta} \overset{\mathrm{T}_0}{\rightarrow} p^{-},\ x^{-} \overset{\mathrm{T}_0}{\rightarrow} p^{\Delta}\right), (\mathrm{C}^{\Delta} \leftrightarrow \mathrm{C}^{-});$$

$$\text{Relative } (R) = \left(x^{\Delta} \overset{\mathrm{T}_9}{\rightarrow} r^{-},\ r^{-} \overset{\mathrm{T}_3}{\rightarrow} x^{\Delta}\right), (\mathrm{C}^{\Delta} \leftrightarrow \mathrm{A}^{-});$$

$$\text{Leading tone exchange } (L) = \left(x^{\Delta} \overset{\mathrm{T}_4}{\rightarrow} l^{-},\ l^{-} \overset{\mathrm{T}_8}{\rightarrow} x^{\Delta}\right), (\mathrm{C}^{\Delta} \leftrightarrow \mathrm{E}^{-}).$$

These elements may be combined to form products of group actions,

$$PR = \left(\mathrm{C}^{\Delta} \overset{P}{\rightarrow} \mathrm{C}^{-},\ \mathrm{C}^{-} \overset{R}{\rightarrow} \mathrm{E}\flat^{\Delta}\right), (\mathrm{C}^{\Delta} \rightarrow \mathrm{E}\flat^{\Delta}).$$

$$RLR = \left(\mathrm{C}^{\Delta} \overset{R}{\rightarrow} \mathrm{A}^{-},\ \mathrm{A}^{-} \overset{L}{\rightarrow} \mathrm{F}^{\Delta},\ \mathrm{F}^{\Delta} \overset{R}{\rightarrow} \mathrm{D}^{-}\right), (\mathrm{C}^{\Delta} \rightarrow \mathrm{D}^{-}).$$

$$RPL = \left(\mathrm{C}^{\Delta} \overset{R}{\rightarrow} \mathrm{A}^{-},\ \mathrm{A}^{-} \overset{P}{\rightarrow} \mathrm{A}^{\Delta},\ \mathrm{A}^{\Delta} \overset{L}{\rightarrow} \mathrm{C}\sharp^{-}\right), (\mathrm{C}^{\Delta} \rightarrow \mathrm{C}\sharp^{-}).$$

The neo-Riemannian group and the *T/I* group are isomorphic to each other and to D_{24}.[50]

1.4. Non-Traditional Triad Usage in a Historical Context

In this section, we investigate non-traditional triadic usage within a specific historical context, and look at various ways triads appear as part of an improvised melodic line. Excerpts from Charlie Parker's improvisations serve as examples to support other authors' claims that non-traditional triadic usage extends well into the bebop era.

---

[49] We hold to the notation where n = the dihedral group's order. Some mathematical texts list the dihedral group where n = number of set elements on which that it acts.

[50] See, Alissa S. Crans, Thomas M. Fiore, and Ramon Satyendra, "Musical Actions of Dihedral Groups," *The American Mathematical Monthly* 116, no.6 (2009): 479-495.

Paul Berliner states that jazz musicians of the 1920s had already begun the development of triadic superimposition through "diatonic upper extensions [tensions] and altered tones of chords." Bassist/composer Rufus Reid describes saxophonist Eddie Harris's use of this technique. "Some [players] conceived of pitch selections as chords superimposed one upon the other—two triads or the polychord type of things…he [Harris] could think real fast that way and superimpose different kinds of harmonic things on the chord because the materials of triads were already second nature and readily at hand."[51] Mark Levine cites Bud Powell as an example of a bebop musician who employed triadic material in the form of slash chords (triad over a bass note) citing as an example Powell's "Glass Enclosure," from the 1953 Blue Note release, *The Amazing Bud Powell, Volume 2*.[52] These accounts support a hypothesis that by the bebop era, a shift toward an increasingly complex musical vocabulary had occurred: one in which included an alternative system of pitch organization that incorporated the superimposition of triads foreign to the composition's stated harmony.

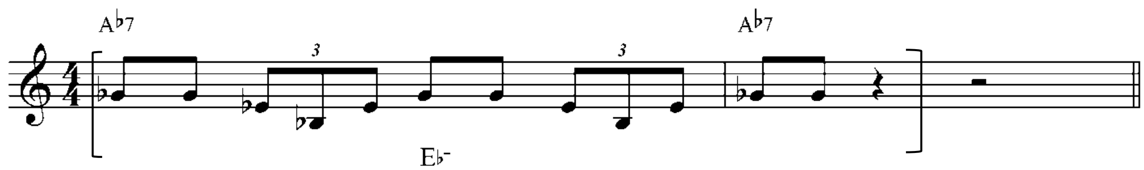
Charlie Parker said that while playing Ray Noble's tune "Cherokee," "I found that by using the higher intervals of a chord as a melody line and backing them with appropriately related changes, I could play the thing I'd been hearing."[53] In the following excerpts from Charlie Parker's solos, we encounter four types of triadic applications, all of which may be considered in terms of melodic superimpositions over stated harmony: (1) relative II$^-$ over a dominant seventh chord;[54] (2) superimpositions based on neo-Riemannian transformations; (3) auxiliary

---

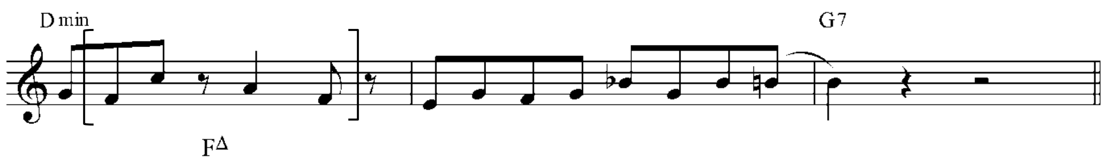
[51] Paul F. Berliner, *Thinking in Jazz: The Infinite Art of Improvisation* (Chicago: The University of Chicago Press, 1994), 160.
[52] Mark Levine, *Jazz Theory* (Petaluma, CA: Sher Music Company, 1995), 110.
[53] Nat Shapiro and Nat Hentoff, *Hear Me Talkin' To Ya* (Mineola, NY: Dover, 1955), 354.
[54] Here the term relative II− does not hold the meaning of a relative minor, but as the subdominant member of a II−—V complex. For example, the relative II$^-$of F^7 is a C$^-$, the relative II$^-$ of A^7 is E$^-$. The II$^-$—V is ubiquitous to jazz harmony. Musicians often base chord/scale determinations around a single II$^-$—V element and apply that single chord/scale choice over both chords, especially when dealing with fast tempos.

superimpositions; and (4) triad chains. In the following examples, stated harmony is positioned above the staff; identification of the superimposed triad is below the staff. [55]

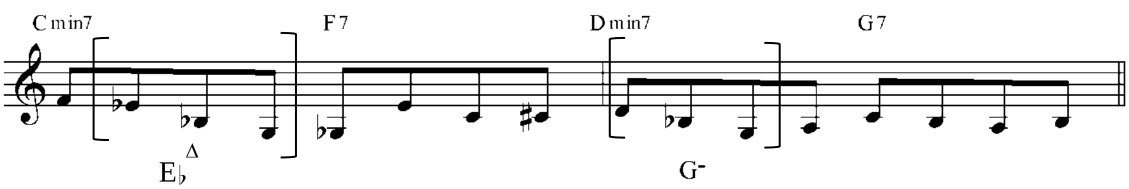
Example 1.1. “Blue Bird,” relative II− over a dominant seventh chord[56]



Example 1.2. “Card Board,” II^-—V complex: relative (*R*) over subdominant harmony, $(\text{D}^- \overset{R}{\leftrightarrow} \text{F}^\Delta)$[57]

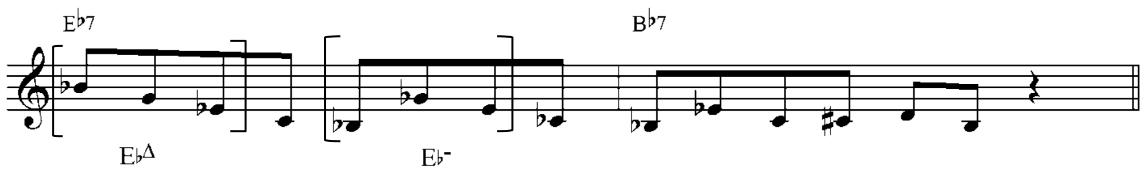


Example 1.3. “Kim” (no.1), II^-—V complex: relative (*R*) over subdominant harmony, $\text{C}^- \overset{R}{\leftrightarrow} \text{E}\flat^\Delta$; parallel (*P*) over dominant harmony $(\text{G}^7 \overset{P}{\leftrightarrow} \text{G}^-)$[58]



[55] Jamey Aebersold, and Ken Sloan, *Charlie Parker Omnibook* (New York: Music People, 1978). Notational use of min for minor in these examples adheres to source nomenclature.
[56] Aebersold and Sloan, 84, mm. 14-5.
[57] Aebersold and Sloan, 94, mm. 29-31.
[58] Aebersold and Sloan, 53, mm. 88-9.

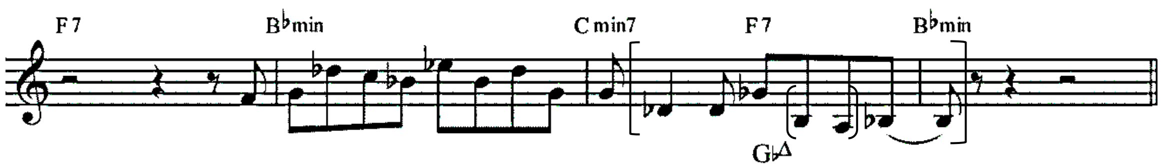
Example 1.4. “Kim” (no.1), parallel (P), ($E\flat^{\Delta} \overset{P}{\leftrightarrow} E\flat^{-}$) [59]



Example 1.5. “Another Hairdo,” leading tone exchange (L), ($B\flat^{\Delta} \overset{L}{\leftrightarrow} D^{-}$)[60]

The following examples of auxiliary superimpositions contain triads that provide greater upper-structure dissonance due to their distant tonal relationship, generating harmonic tensions.

Example 1.6. “Diverse,” II^{-}—V complex: $\flat\mathrm{II}^{\Delta}$ over dominant harmony[61]



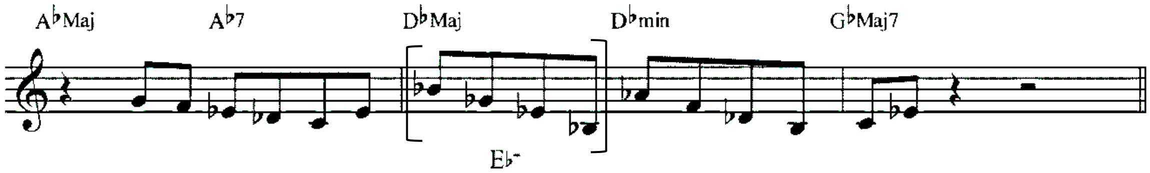
Example 1.7. “The Bird,” II^{-}—V complex: $\flat\mathrm{II}^{-}$ over dominant harmony[62]

---

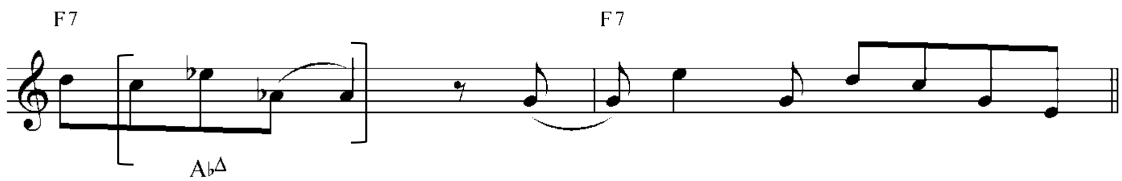
[59] Aebersold and Sloan, 53, mm. 44-5. The above example could read as $C^{-} \rightarrow C\flat^{\Delta}$, the Riemannian Slide (S) transformation where the two triads share a common third.

[60] Aebersold and Sloan, 105, mm. 54-5.
[61] Aebersold and Sloan, 114, mm. 25-8.
[62] Aebersold and Sloan, 111, mm. 35-7.

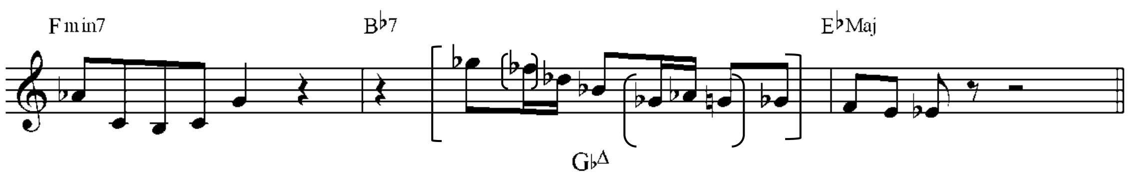
Example 1.8. “Bird Gets the Worm,” II^{-} over a major seventh chord[63]



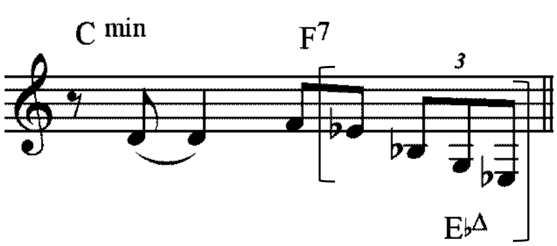
Example 1.9. “Vista,” ♭III$^{\Delta}$ over a dominant seventh chord[64]



Example 1.10. “Warming up a Riff,” II^{-}—V complex: ♭VI$^{\Delta}$ over dominant harmony[65]

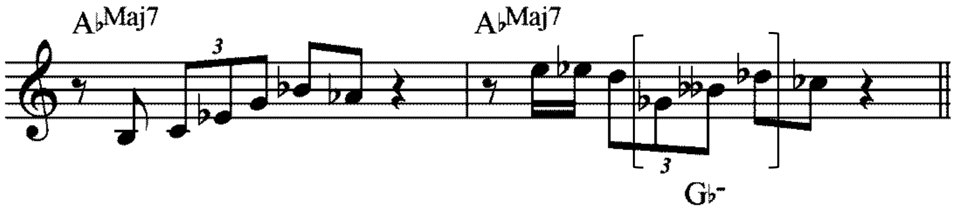


Example 1.11. “Another Hairdo,” II^{-}—V complex: ♭VII$^{\Delta}$ over dominant harmony[66]



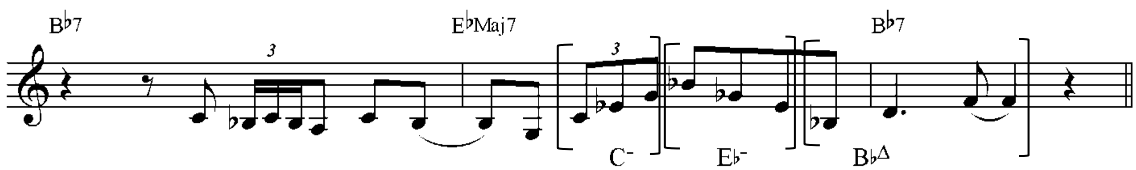
[63] Aebersold and Sloan, 94, mm.13-5.
[64] Aebersold and Sloan, 100, mm. 6-7.
[65] Aebersold and Sloan, 136, mm.19-21.
[66] Aebersold and Sloan, 105, m. 60.

Example 1.12. "Klaun Stance," $\flat VII^-$ over a major seventh chord[67]



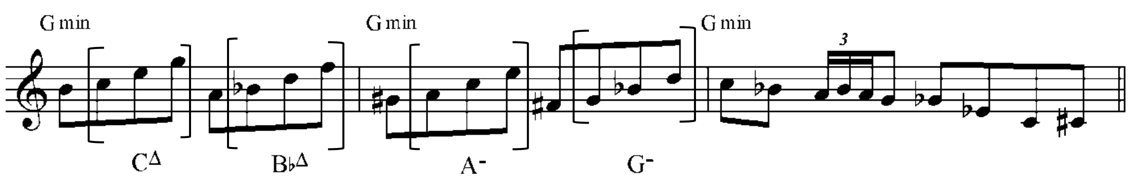
Triad chains consist of two or more triads in succession. In the following three examples, their orbits are of order > 2.

Example 1.13."Mohawk" (no.1), triad chain (C^-, $E\flat^-$, $B\flat^\Delta$)[68]



Parker unfolds a C blues scale with an added ninth: $C^- = \{C,E\flat,G\}$; $E\flat^- = \{E\flat,G\flat,B\flat\}$; $B\flat^\Delta = \{B\flat,D,F\}$, where $C^- \cup E\flat^- \cup B\flat^\Delta = \{C,D,E\flat,F,G\flat,G,B\flat\}$. Let C = $\hat{1}$. Therefore, the pitch content expressed as scale degrees equals ($\hat{1}$, $\hat{2}$,$\flat\hat{3}$, $\hat{4}$,$\flat\hat{5}$,$\natural\hat{5}$,$\flat\hat{7}$).

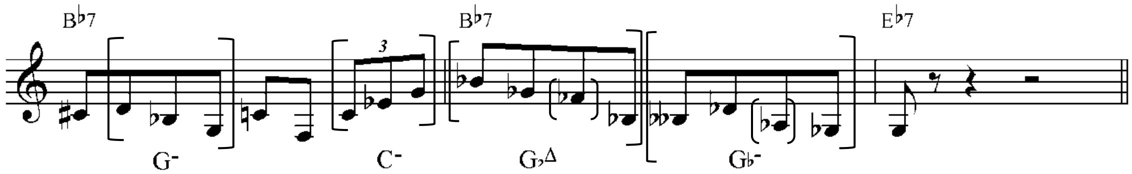
Example 1.14. "Ah-Leu-Cha," triad chain (C^Δ, $B\flat^\Delta$, A^-, G^-)[69]



Triadic material in this example consists of diatonic chords obtained from the (1♭) diatonic; triadic root motions unfold a descending diatonic step-wise progression. The melodic gesture is constructed from root-position triads and a prefix incomplete lower neighbor to the triad's root.

[67] Aebersold and Sloan, 90, mm. 41-2.
[68] Aebersold and Sloan, 38, mm.13-5.
[69] Aebersold and Sloan, 88, mm. 65-7.

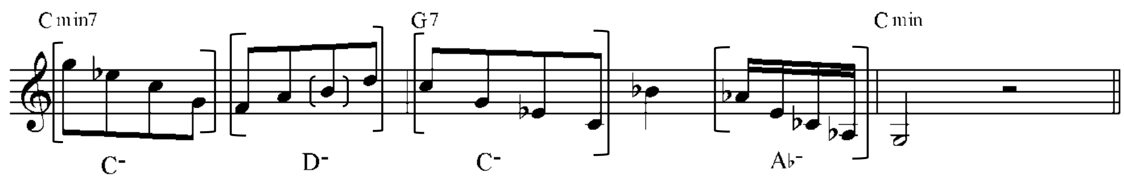
Example 1.15. "Bloomdido," triad chain (G^-, C^-, $G\flat^{\Delta}$, $G\flat^-$)[70]



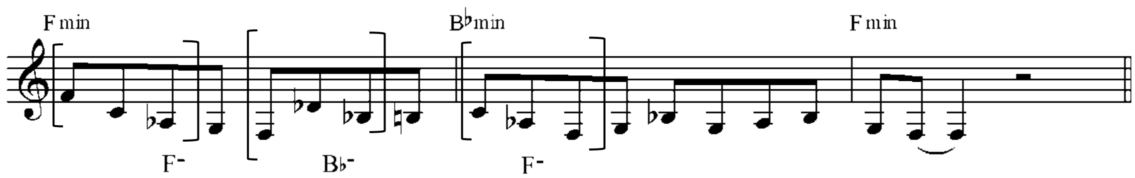
G^- acts as a minor-inflected dominant to C^-. C^- attempts to move to its T_6 image. The $G\flat^{\Delta}$ intercalation, through *P*, immediately corrects its parity to ($G\flat^-$).

The next two examples feature triad chains that contain orbits of order 2.

Example 1.16. "The Bird," triad chain containing a triad pair, ((C^-, D^-), $A\flat^-$)[71]



Example 1.17. "Bird Gets the Worm," triad pair, (F^-, $B\flat^-$)[72]



A functional reading of the above two examples is quite rudimentary. D^- in Example 1.16 acts as a complete upper neighbor to the stated C^-. In Example 1.17, $B\flat^-$ is a minor inflected subdominant, acting as a harmonic anticipation of the stated harmony in the second measure. A permutational reading of the last two examples identifies the triad pairs as group actions, with orbits of order 2, (C^-, D^-) and (F^-, $B\flat^-$).

[70] Aebersold and Sloan, 109, mm. 39-41.
[71] Aebersold and Sloan, 111, mm.55-7.
[72] Aebersold and Sloan, 95, mm. 33-5.

Definition 11. Involution

An involution is an operation of order 2, regardless of the number of elements of a set that it permutes, shown with the mapping

$$f(f(x)) = x, f^2(x) = x,$$

written cyclically as (x, y).

The set of neo-Riemannian transformations P, L, and R contains three involutions. Written as actions on C^{Δ}, we have the following: $\{(C^{\Delta} \overset{R}{\leftrightarrow} A^{-}), (C^{\Delta} \overset{P}{\leftrightarrow} C^{-}), (C^{\Delta} \overset{L}{\leftrightarrow} E^{-})\}$. The twelve inversion operations (I_n) described in set theory also form a set of involutions. In mathematics, a *transposition* (or 2-cycle) is an exchange of (only) two elements in a set. Because of the potential confusion of the mathematical term transposition and the musical term of the same name (which mathematicians would call *translation*), we will adopt the term "exchange" for all mathematical transpositions.[73] We say the group member (C^{-}, D^{-}) is an exchange generated by some action f on a set containing C^{-} and D^{-}.

[73] See, Hook, 97.

CHAPTER 2. SET DEFINITION

2.1. Introduction

This section contains an overview of established jazz harmonic theory, chord/scale relationships and improvisational methods, which are rooted in functional harmonic practices, and which ultimately serve as criteria for set definition. While the central argument of the Permutational Triadic Approach pertains to the relationships between consonant triads generated by permutation groups, this does not mean that the permutations exist within a tonal vacuum. This speculative theory respects the functional underpinnings of tonal theory, and expands upon accepted theoretical practice. Therefore, functional harmony, to the degree to which it informs the chord/scale relationship, is the basis for set definition.

There exist numerous accepted approaches to chord/scale determination. One widely employed chord/scale method, attributed to David Baker, and published by Jamey Aebersold, considers chord quality paramount in determining a scale choice.[74] However, if one considers chord quality as the only criterion for chord/scale determination, *Mehrdeutigkeit* (multiple meaning), an important aspect of functional harmony in general, remains unaddressed. Brian Hyer addresses this concept in his discussion on the use of Roman numerals as an analytical tool:[75]

---

[74] Aebersold, *Jazz Handbook*, 14.

[75] *Mehrdeutigkeit* is a term first used to explain the "multiple meaning" of musical objects and was first used to describe nineteenth-century music. The term is attributed to theorist Gottfried Weber. See, Gottfried Weber, *Versuch einer geordneten Theorie der Tonsetzkunst*, 3rd ed. (Mainz: B. Schott, 1830-2); Janna K. Saslaw, "Gottfried Weber's Cognitive Theory of Harmonic Progression," *Studies in Music from the University of Western Ontario* 13 (1991): 121-44; "Gottfried Weber and Multiple Meaning," *Theoria* 5 (1990-1): 74-103; "Gottfried Weber and the Concept of *Mehrdeutigkeit*" (PhD. diss., Columbia University, 1992); David Carson Berry, "The Meaning of "Without": An Exploration of Liszt's Bagatelle ohne Tonart," *19th-Century Music* 29, no.3 (2004): 230-62.

> A recurring source of vexation in scale-degree theories is *Mehrdeutigkeit* , or multiple meaning: because harmonies assume roman numerals on the basis of pitch-class content rather than musical behavior (as in function theories), there are no hard and fast criteria to determine which major or minor scale a particular harmonic configuration refers to: a C major triad, for instance, can be heard as I in C major, IV in G major, V in F major, or VI in E minor; one must take contextual factors into account in order to narrow down the possibilities to a single roman numeral.[76]

In order to address *Mehrdeutigkeit*, we use a system of chord/scale determination that takes into account the chord's function as well as its quality.[77] For the present discussion, functional harmony is divided into five subcategories: diatonic harmony; dominant action; modal harmony, including modal interchange (modal mixture); and tonic systems.

2.2. Diatonic Harmony

Functional diatonic harmony in jazz is well described by Hugo Riemann's three functional classes: tonic, subdominant, and dominant. Riemann calls these classes "primary pillars of harmonic progression,"[78] describing I, IV, V as primary tonic, primary subdominant and primary dominant chords respectively. The remaining chords function as secondary harmonies (not to be confused with secondary dominants) belonging to a class associated with a primary chord: tonic {I^{Δ}, VI^{-}, III^{-}}; subdominant {IV^{Δ}, II^{-}, in some cases VI^{-}}; and dominant {V^{Δ}, VII^{o}, in some cases III^{-}).[79] Jazz harmonic theory retains many of these Riemannian

[76] Brian Hyer, "Tonality," in *The Cambridge History of Western Music Theory*, ed. Thomas Christensen (Cambridge: Cambridge University Press, 2002), 726-52.
[77] See Barrie Nettles and Richard Graf, *The Chord Scale Theory and Jazz Harmony* (Rottenberg: Advance Music, 1997).

[78] William C. Mickelsen, trans. and ed., *Hugo Riemann's Theory of Harmony: A Study by William C. Mickelsen and History of Music Theory, Book III by Hugo Riemann* (Lincoln: University of Nebraska Press, 1977), 28-9. See also, Hugo Riemann, *Handbuch der Harmonielehre*, 9th ed. (Liepzig: Breitkoph und Härtel, 1921), 215.
[79] Hugo Riemann, *Präludien und Studien*, vol.3, (Leipzig, 1901), 4.

concepts; differences are the omission of VI^{-7} from the subdominant class, and the omission of III^{-7} and $\mathrm{VII}^{-7\flat 5}$ from the dominant class.[80]

2.3. Modal Harmony

Modal theory describes the cyclic reordering of the pitch classes of a nonsymmetrical scale and the assignment of a new tonic, isomorphic to the set of permutations generated by rotational symmetry. The cyclic group of degree 7, C_7, has an action on the diatonic collection's seven pitches as an ordered set in register. The seven elements may represent diatonic pitches, triads or seventh chords.

Definition 12. Cyclic group

A group G is cyclic if there exists an element r in G that holds to the group presentation,[81]

$$G := \{r^n \mid n \in \mathbb{Z}, r^n = i\}.$$

Element r is called the generator of G, written $G = \langle r \rangle$. The set of integers $\mathbb{Z}_n$ under addition mod n is cyclic; cyclic groups are the only groups that can be generated by a single element.

$$\underbrace{1 + 1 + \cdots + 1}_{|n| \text{elements}}.$$ [82]

If $G := \{r^i, r^1, r^2, r^3, r^4\}$ and G is a group, then $r^5 = r^i$, G is cyclic, isomorphic to the set of integers $\{0,1,2,3,4\}$ under addition modulo 5: $r^1 \cdot r^2 = r^3$; $r^2 \cdot r^4 = r^6 = r^1$. For every positive integer n, there exists one cyclic group of order n, describing n-fold rotational symmetry.[83] Cyclic groups

[80] In jazz functional harmony, $\mathrm{VII}^{-7\flat 5}$ is rarely considered a dominant alias; it is most often a relative II^{-} of $\mathrm{V}^7/\mathrm{VI}^{-}$.

[81] The group presentation reads as follows, the first set of entries is the group generators, followed by | (meaning "such that"), and concluding with necessary and sufficient relations on the generators to define the group. In this case, G is the group generated by r, such that, r^3 returns the group identity element, i.

[82] Adapted from Gallian, 73-4.

[83] We have seen previous examples of cyclic groups. Figure 2, showing the full symmetry group of the triangle, contains C_3, the rotations on the triangle as a subgroup.

are the simplest of groups and are often the building blocks of more complex groups.[84] Define the set $\boldsymbol{D}_{(\emptyset)}$:= {1=C, 2=D, 3=E, 4=F, 5=G, 6=A, 7=B},[85] and let $J := (\boldsymbol{D}_{(\emptyset)}, C_7)$ be a group.

Example 2.1. Diatonic modes modeled as C_7

$$\begin{array}{cc} \text{Ionian} & i \\ \text{(Locrian)} & r \\ \text{Aeolian} & r^2 \\ \text{Mixolydian} & r^3 \\ \text{Lydian} & r^4 \\ \text{Phrygian} & r^5 \\ \text{Dorian} & r^6 \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \\ 5 & 6 & 7 & 1 & 2 & 3 & 4 \\ 6 & 7 & 1 & 2 & 3 & 4 & 5 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

Example 2.2. C_7 in cyclic notation

$$\begin{array}{ccc} \text{Ionian} & i & (1)(2)(3)(4)(5)(6)(7) = (\text{C})(\text{D})(\text{E})(\text{F})(\text{G})(\text{A})(\text{B}) \\ \text{Dorian} & r & (2176543) = (\text{DEFGABC}) \\ \text{Phrygian} & r^2 & (3164275) = (\text{EFGABCD}) \\ \text{Lydian} & r^3 & (4152637) = (\text{FGABCDE}) \\ \text{Mixolydian} & r^4 & (5147362) = (\text{GABCDEF}) \\ \text{Aeolian} & r^5 & (6135724) = (\text{ABCDEFG}) \\ \text{Locrian} & r^6 & (7123456) = (\text{BCDEFGA}) \end{array}$$

In the example, Ionian represents the diatonic triads as the identity element.[86] To identify a specific mode, one locates the permutation in which the modal scale degree maps to 1, creating a new tonic. For example, Dorian is generated when the Ionian's $\hat{2}$ becomes the new $\hat{1}$ (tonic). The reading for Dorian in cyclic notation is as follows: 2 maps to the place previously held by 1; 1 maps to the place previously held by 7, 7 maps to the place previously held by 6, and so on.

[84] For a detailed study of scales defined as cyclic groups see Paul F. Zweifel, "Generalized Diatonic and Pentatonic Scales: A Group-Theoretic Approach," *Perspectives of New Music* 34, no.1 (1996): 140-161.
[85] A complete description of the twelve diatonic collections in chromatic space requires twelve copies of *J*.
[86] This follows the musical practice of relating modes back to the parent Ionian as a baseline to model scale degrees; e.g. Dorian contains scale degrees ♭3 and ♭7 by comparison to its parallel Ionian.

To identify a specific modal permutation, locate the orbit where the modal-tonic maps to 1. Integer presentation in cyclic notation traditionally situates the smallest-value integer in the orbit's first position. For musical relevancy, however, the elementwise orderings in Example 2.2 place the modal tonic scale degree in the left-most entry. Diatonic modes not only share equivalent pitch class content, they are isomorphic under the group action ($\boldsymbol{D}_{(n)}$, C_7).

The aural phenomenon created by modes relies upon this pitch content reordering; how musicians realize the reordering harmonically varies. Ron Miller uses the location of half-steps within the parent collection as the definitive criteria. This technique is quite useful when dealing with non-tertian collections. The method used in this document adheres to the following steps: (1) Tonic is represented by a new scale degree. (2) Scale degree $\hat{3}$ above the new tonic acts as a mode quality identifier (any diatonic mode is some version of either major or minor). (3) Characteristic pitches are scale-degree inflections unique to a particular mode: ($\natural\hat{6}$) Dorian, ($\flat\hat{2}$) Phrygian, ($\sharp\hat{4}$) Lydian, ($\flat\hat{7}$) Mixolydian, and ($\flat\hat{6}$) Aeolian.

Modal harmonic theory utilizes four functional classes: tonic (**T**), characteristic (**C**), avoid (**A**), and passing (**P**). Tonic chords are as defined; they represent the modal tonic. Characteristic chords contain characteristic scale degrees, thereby conveying the mode's sonority. Due to an internal tritone, avoid chords tend to destroy the sound of the mode by implying an expected resolution to the parent Ionian rather than the modal tonic; however, there are instances where the triadic form of an avoid chord is allowed as a characteristic chord, as is the case of II^{Δ} (II^{7}) in Lydian. Passing chords act as connective harmonies.[87]

To model modal harmony as C_n, simply replace pitch/integer correspondences with Roman numeral/integer correspondences. Retaining the 0♯/0♭ diatonic assumption, define the set

[87] The source of this method is an unpublished document acquired by the author at Berklee College of Music in 1987. This was a handout provided to the Advanced Modal Harmony Class by instructor Steve Rochinski. See also, Nettles and Graff, 128-35.

$\boldsymbol{K}$:= {1 = $C^{\Delta 7}$, 2 = D^{-7}, 3 = E^{-7}, 4 = $F^{\Delta 7}$, 5 = G^{7}, 6 = A^{-7}, 7 = $B^{-7\flat 5}$}, and the group $H := (\boldsymbol{K}_{(\emptyset)}, C_7)$. Geometrically, a regular n-gon with n elements (which could be the sides, edges, or vertices) models n-fold rotational symmetry, $\boldsymbol{C}_n$.[88] Since the diatonic collection, with its seven elements, is presently the topic of discussion, we use a regular septagon as the modeling agent. Figures 3 and 4 contain geometric representations of 0♯/0♭ Ionian (i) and 0♯/0♭ Dorian (r).

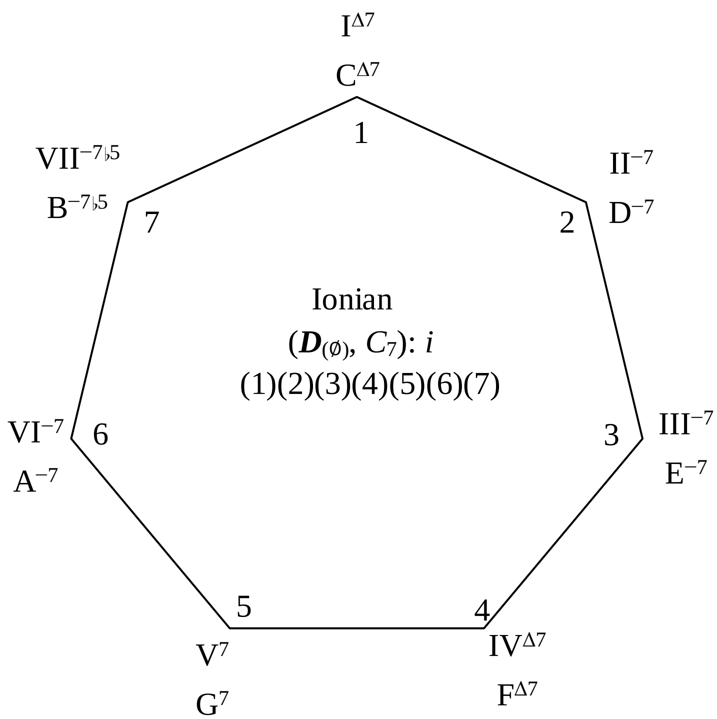


Figure 3. Ionian as $(\boldsymbol{D}_{(\emptyset)}, C_7)$: i

[88] Points plotted upon a circle are not acceptable as circle group elements are uncountable (the circle has no vertices) and cyclic group elements must be countable.

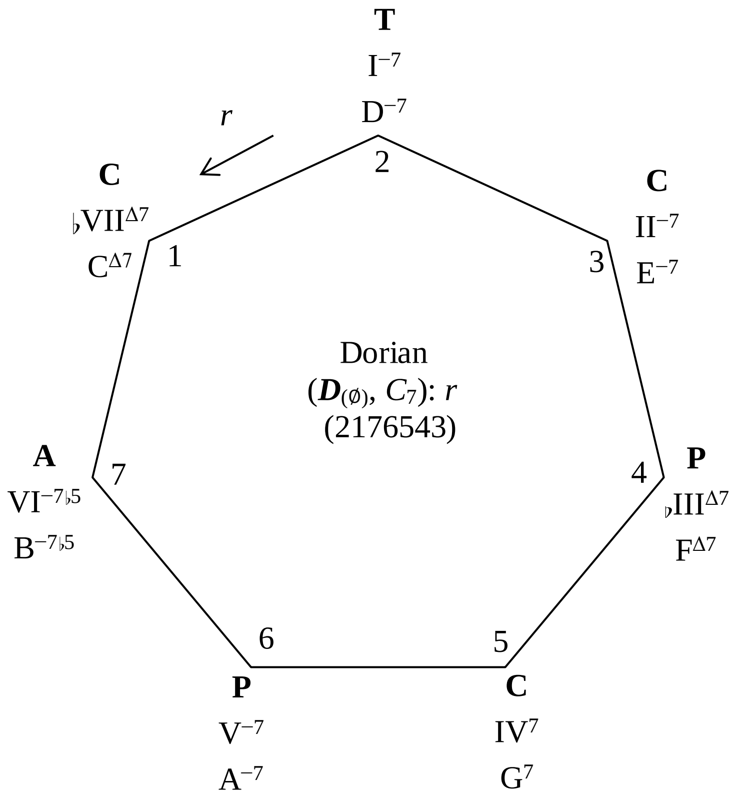


Figure 4. Dorian as ($\boldsymbol{D}_{(\varnothing)}$, C_7): *r*

Appendix A displays harmonic analyses for the remaining diatonic modes.[89]

All chords from the parallel Aeolian and any characteristic chord from the remaining parallel modes qualify as modal interchange harmonies; therefore, in the key of C major, the set of all available modal interchange chords is defined as all chords from C Aeolian and the characteristic chords from C Dorian, C Phrygian, C Lydian, and C Mixolydian. Example 3 lists all available modal interchange chords as a modal interchange array. Aeolian-derived chords provide modal inflections to the Ionian key areas: tonic = I^{-7}, $\flat\mathrm{III}^{\Delta 7}$, dominant = V^{-7}, $\flat\mathrm{VII}^{7}$. Subdominant key area modal interchange chords form a distinct harmonic class called

[89] Locrian is excluded due to the instability of its tonic chord. In jazz, $\mathrm{VII}^{-7\flat 5}$ functions primarily as a passing chord between VI^{-} and I^{Δ} or as the relative $\mathrm{II}^{-7\flat 5}$ to either $\mathrm{V}^{7}/\mathrm{VI}^{-}$ or ${}^{\mathrm{sub}}\mathrm{V}^{7}/\mathrm{VI}^{-}$.

subdominant minor, which includes II$^{-7\flat5}$, IV^{-} (including IV^{-7} and IV^{-6} variants), $\flat$VI$^{\Delta7}$, and $\flat$II$^{\Delta7}$.

Example 3. Modal interchange array

I^{-7}
Aeolian

VII^{-7}
Lydian

$\flat$II$^{\Delta7}$
Phrygia
n

$\flat$VII^{-7}
Phrygian

II^{7}
Lydian
(triad)

$\flat$VII^{7}
Mixolydian
Aeolian

II^{-7}
Dorian

I$^{\Delta}$
GenericTonic
(Major Key)

$\flat$VII$^{\Delta7}$
Dorian
Mixolydian

II$^{-7\flat5}$
Aeolian

$\flat$VI$^{\Delta7}$
Aeolian

$\flat$III$^{\Delta7}$
Aeolian

V^{-7}
Mixolydian
Aeolian

IV^{7}
Dorian
(triad)

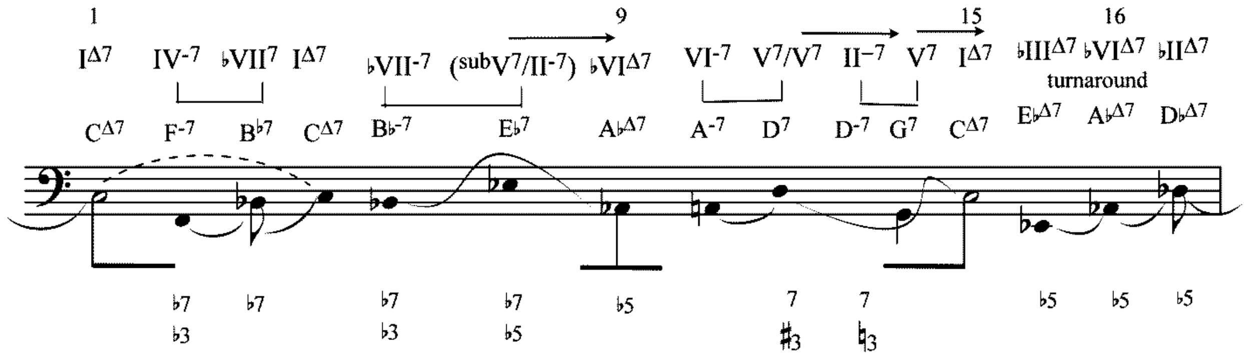
V$^{\Delta7}$
Lydian

IV^{-7}
Aeolian

Analysis 1. "Lady Bird"

Modal interchange can operate on multiple harmonic levels simultaneously within a single composition, as in Tadd Dameron's "Lady Bird."

Example 4. "Lady Bird," graphic analysis



In "Lady Bird," modal interchange informs a large-scale structural arrival of $\flat VI^{\Delta 7}$ in m.9, and a local harmonic event, the turnaround, which propels the music back to the top of the form. The turnaround features three modally inflected chords taken from two modes, $\flat III^{\Delta 7}$ [Aeolian], $\flat VI^{\Delta 7}$ [Aeolian], and $\flat II^{\Delta 7}$ [Phrygian].

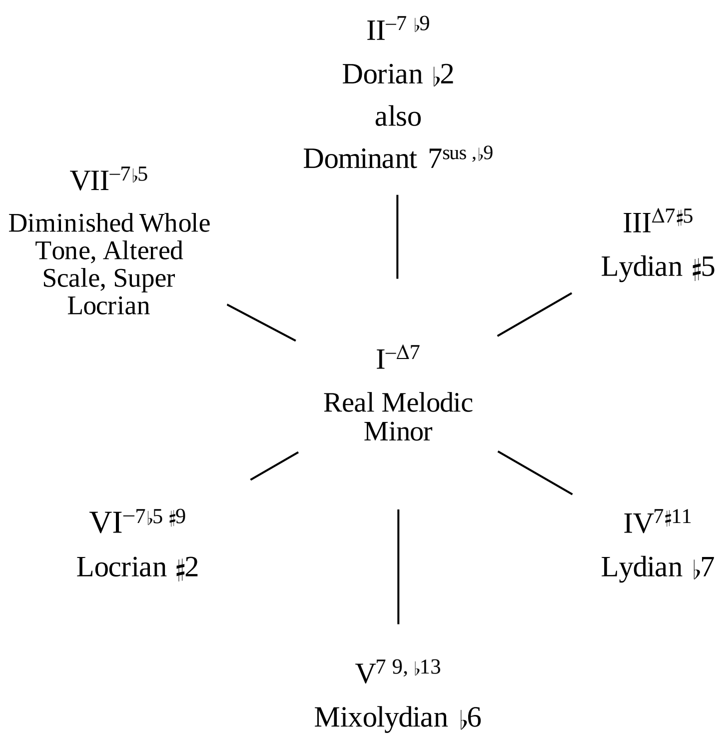
Generation of modal elements though rotational permutation also applies to scale collections other than the diatonic, including, what is termed here, synthetic scales.[90] The synthetic scales included in this study include: real melodic minor, real melodic minor ♯5, harmonic minor, harmonic major, double harmonic $\{\hat{1}\flat\hat{2},\hat{3},\hat{4},\hat{5},\flat\hat{6},\hat{7}\}$ and double harmonic ♯5 $\{\hat{1}\flat\hat{2},\hat{3},\hat{4},\sharp\hat{5},\hat{6},\hat{7}\}$.[91] The difference in the musical usage of the diatonic modes versus synthetic modes is that synthetic scales generally do not form a composition's harmonic foundation. Modes from synthetic scales do, however, act as local reharmonizations, generate modal interchange chords within a diatonic harmonic progression, and generate chord/scale possibilities. In the following analysis, inclusions of modally derived harmonies from synthetic scales define the composition's harmonic areas.

[90] Ron Miller calls these collections "altered diatonic scales." See Miller, 31-35, 89-93, 115-18.
[91] A musical representations of these scales are presented in Table 3, the Scale Roster.

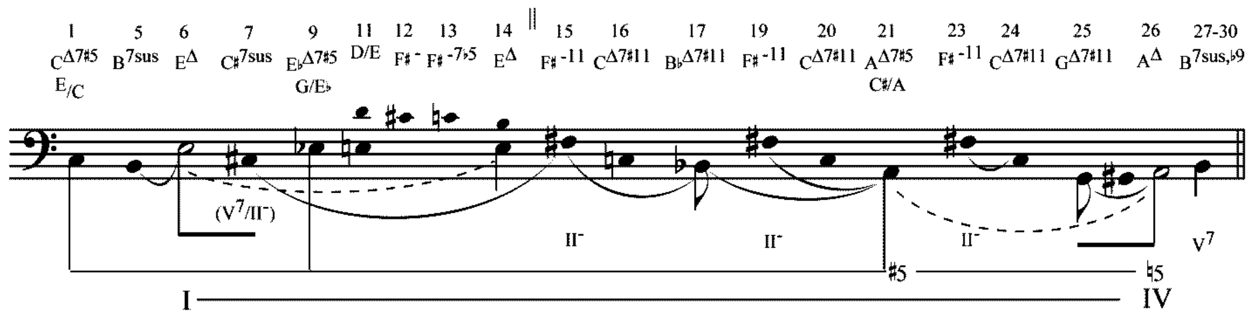
Analysis 2. “The Beatles”

John Scofield’s “The Beatles” exhibits how diatonic, hexatonic, and real melodic minor harmonic sonorities can intermingle within a single composition. As a preliminary to the analysis, Example 5 lists the modal representations of the real melodic minor. Modal representations of the remaining synthetic scales used in this study are in Appendix A.2. Appendix B.1 contains an annotated lead sheet for “The Beatles.” Example 6 contains a graphic analysis.

Example 5. Modal representation of the real melodic minor



Example 6. “The Beatles,” graphic analysis



The analysis addresses each scale genre separately and describes its role within the composition.

(1) Diatonic: A four-sharp diatonic, E major, is the prevailing tonal center where the tonic (E^{Δ}) makes an arduous journey to attain the subdominant (A^{Δ}); this tonic-to-subdominant motion forms the harmonic background. Measures 5-6 introduce the tonic sonority, with an initial tonic arrival in m. 6, followed by an unresolved (at least in the immediate sense) V^7/II^- ($C\sharp^{7\ sus}$) in m.7. Measures 12-13 unfold subdominant harmonies, one of which includes an internal chromatic line that involves the subdominant alias II^- and its modal interchange counterpart $II^{-7\flat5}$ over a tonic pedal. Measure 15 ushers in the binary form’s second section through the restatement of II^{-7}, followed by the subdominant-minor chord $\flat VI^{\Delta7}$, moving to a $B\flat^{\Delta7}$ that functions as an upper neighbor to a modally inflected $IV^{\Delta7\sharp5}$ ($A^{\Delta7\sharp5}$). Measures 19-22 contain a second attempt to attain the true subdominant, where the arrival of $A^{\Delta7\sharp5}$ provides the correct pitch level. However, modally inflected chord quality thwarts true subdominant attainment. Measures 23-26 start with a false recapitulation of the second section, making a third and final attempt for IV^{Δ}. The harmonic background closes in m. 26, with the long awaited arrival of IV^{Δ}

(A^{Δ}) in its simple triadic form. The final four measures act as a turnaround on $B^{7\ sus\flat 9}$, which is an altered form of V^7/E^{Δ}.

(2) Hexatonic: The opening chord, $C^{\Delta 7\sharp 5}$, attempts to be two things at once, the initial statement of tonic (the tonic triad is the upper-structure triad) and an altered form of a $\flat VI^{\Delta 7}$, a subdominant-minor chord. Nevertheless, aurally, it is an off-tonic opening affected by a *double modal interchange* chord. A double modal interchange chord derives it's root from a parallel mode and a differing scale genus provides the quality. For an example, let us assume the key of E major and consider a chord built on ♭VI. E Aeolian provides the root (C), and to continue deriving the modal interchange quality from Aeolian, we would say, $\flat VI^{\Delta 7}$. However, to include an additional modal interchange component, choose the modal interchange chord's quality from a differing scale genre. Major seventh ♯5 is a chordal quality available in real melodic minor (it is the third mode). To conjoin the major seventh ♯5 quality (from real melodic minor) with ♭VI root presentation (from Aeolian), we generate $C^{\Delta 7\sharp 5}$, a double modal interchange chord in the key of E major.

Major seventh ♯5 chords appear three times within the composition, where each subsequent presentation lays T_3 above the previous presentation (see m.9 ($E\flat^{\Delta 7\sharp 5}$)) and mm. 21-22 ($A^{\Delta 7\sharp 5}$)), the latter being the problematic subdominant resolution. These chords also try to be two things at once in terms of their parent scale genus. One reading describes them as a modal variant of the real melodic minor, taking into account the brief utterance of the pitch A in the bass line of m. 2. $C^{\Delta 7\sharp 5}$, the chord in m.2, can derive from the third mode of the real melodic minor built in the pitch A; however, the hexatonic also generates major seventh ♯5. The descending arpeggiation (B,G♯,E♯,C♯,A,G♯,E♯,C♯), in the third the major seventh sharp five occurrence, $A^{\Delta 7\sharp 5}$, in mm. 21-22, when taken in its entirety, unfolds a subset of the real melodic minor built on F♯.

Realize the initial pitch, B, in m.21 as a harmonic tension (9^{th}) over the $A^{\Delta 7\sharp 5}$ (it does not return as the arpeggiation continues in m.22), thus removing it from the descending arpeggiation proper. The set of pitches {G♯,E♯,C♯,A,G♯,E♯,C♯} remain, opening the possibility of reading the descending arpeggiation as a $Hex_{(0,1)}$ subset. Given that the major seventh ♯5 chords move according to a symmetrical transpositional level, (T_3), a reading that ties additional symmetrical structures into the analysis supports a more general hypothesis that symmetric structures bolster other symmetric structures.

The hexatonic reading considers three unique hexatonic collections to be at play: $C^{\Delta 7\sharp 5} \in Hex_{(3,4);}$ $E\flat^{\Delta 7\sharp 5} \in Hex_{(2,3)}$; and $A^{\Delta 7\sharp 5} \in Hex_{(0,1)}$. It is worth pointing out an interesting property. The chords ascend in chromatic-space, $C^{\Delta 7\sharp 5} \xrightarrow{T_3} E\flat^{\Delta 7\sharp 5} \xrightarrow{T_3} A^{\Delta 7\sharp 5}$ and simultaneously descend in hexatonic space, $Hex_{(3,4)} \rightarrow Hex_{(2,3)} \rightarrow Hex_{(0,1)}$.

(3) Real melodic minor: The analytical decision to include $C^{\Delta 7\sharp 5}$, $E\flat^{\Delta 7\sharp 5}$, and $A^{\Delta 7\sharp 5}$ as hexatonic subsets isolates them from the turn-around harmony, $B^{7sus,\flat 9}$, a typical real melodic minor chord. As such, the final sonority, $B^{7sus,\flat 9}$, clearly conveys the second mode of the real melodic minor and is not related to the hexatonic chords through a generative parent scale. $B^{7sus,\flat 9}$ functions as an altered V^7 of E^{Δ}. Note that the final sonority could present as an upper-structure triad over a bass note, A^{-}/B, where a modally inflected IV^{-} chord resides in the upper-structure, further substantiating the importance of the subdominant and modal interchange in the composition.

The following discussion applies modal interchange theory as a component of set definition. For illustration, consider the progression II^{-7}—V^7—$I^{\Delta 7}$ in the key of C major (D^{-7}—G^7—$C^{\Delta 7}$). Referring back to the musical domains mentioned in the introduction, there are two possible approaches to determining a chord/scales for this progression. First, since all the harmonies derive from the 0♯/0♭ diatonic, one could choose to play C Ionian scale over all three chords—thus defining the musical domain as C Ionian—and pull consonant triads from that collection to define the set. Alternatively, the musician could choose to address each chord separately, and apply D Dorian over D^{-7}, G Mixolydian over G^{7}, and C Ionian over $C^{\Delta 7}$. Although we have not left the realm of C Ionian in the latter approach, there are three scales to manipulate.

The inclusion of modal interchange adds variety and a more complex soundscape. Example 7 contains a reconsideration of the II—V—I progression using modal interchange and double modal interchange. The chord scale choices are listed in two forms to clearly show the scale genre from which the scale is derived: common tone, where the pitch level of the scale remains invariant, and parallel, where the name of the chord/scale agrees with the harmonic presentation.[92] In Example 7.1, $D\flat^{\Delta 7}$, a characteristic chord from C Phrygian, acts as a passing chord between the subdominant and tonic members, replacing the G^{7}. With this replacement, an additional scale, C Phrygian, allows set determination to derive from (0♯/0♭) and (4♭) diatonic collections.

---

[92] See, Mick Goodrick, *The Advancing Guitarist* (Milwaukee: Hal Leonard, 1987), 62-7.

Example 7.1. Modal interchange as set determinant, II—V—I

Analysis	II^{-7}	V^{7}	$I^{\Delta 7}$
Stated Harmony	D^{-7}	G^{7}	$C^{\Delta 7}$
Chord/Scale: Common tone	C Ionian	C Ionian	C Ionian
Chord/Scale: Parallel	D Dorian	G Mixolydian	C Ionian
Analysis	II^{-7}	$\flat II^{\Delta 7}$	$I^{\Delta 7}$
Reharmonization	D^{-7}	$D\flat^{\Delta 7}$	$C^{\Delta 7}$
Chord/Scale: Common tone	C Ionian	C Phrygian	C Ionian
Chord/Scale: Parallel	D Dorian	D♭ Lydian	C Ionian

Example 7.2. Modal Interchange as set determinant, I—III—VI

Analysis	$I^{\Delta 7}$	III^{-7}	VI^{-7}
Stated Harmony	$C^{\Delta 7}$	E^{-7}	A^{-7}
Chord/Scale: Common tone	C Ionian	C Ionian	C Ionian
Chord/Scale: Parallel	C Ionian	E Phrygian	A Aeolian
Analysis	$I^{\Delta 7}$	$\flat VII^{\Delta 7}$	VI^{-7}
Reharmonization	$C^{\Delta 7}$	$B\flat^{\Delta 7}$	A^{-7}
Chord/Scale: Common tone	C Ionian	C Mixolydian	C Ionian
Chord/Scale: Parallel	C Ionian	B♭ Lydian	A Aeolian

In Example 7.2, $B\flat^{\Delta 7}$, a characteristic chord from C Mixolydian, acts as passing chord between the true tonic and a tonic alias and replaces the tonic alias III^{-7}, breaking up the saturation of three consecutive tonic harmonies.

The next example uses a II^{-7}—V^7—$I^{\Delta 7}$ progression in D♭ major as it is found in Kenny Dorham's composition "Blue Bossa." The composition is in key of C minor and the harmonic content consists of a II—V—I progression in the home key followed by the II—V—I that tonicizes ♭II^{Δ} (D♭$^{\Delta}$). It then returns to a II—V—I in the home key.

Example 7.3.Synthetic scale modal interchange as set determinant

Analysis	II^{-7}	V^7	$I^{\Delta 7}$
Stated Harmony	E♭$^{-7}$	A♭^{7}	D♭$^{\Delta 7}$
Chord/Scale: Common tone	D♭ Ionian	D♭ Ionian	D♭ Ionian
Chord/Scale: Parallel	E♭ Dorian	A♭ Mixolydian	D♭ Ionian
Analysis	II^{-7}	V^7	$I^{\Delta 7\sharp 5}$
Reharmonization	E♭$^{-7}$	A♭^{7}	D♭$^{\Delta 7\sharp 5}$
Chord/Scale: Common tone	D♭ Ionian	D♭ Ionian	B♭ real melodic minor
Chord/Scale: Parallel	E♭ Dorian	A♭ Mixolydian	D♭ Lydian Augmented

The melodic minor provides for a scale genus other than diatonic, a quality missing in the previous examples. In addition, the opening C^{-} takes Aeolian as a first-choice chord/scale but C Dorian is also a viable possibility. The introduction of the pitch A♮ is facilitated by D♭$^{\Delta 7\sharp 5}$ foreshadows the A♮ contained in C Dorian if indeed the musician desires to incorporate Dorian-inflected material.

The final example in this section draws upon certain aspects from the discussion of the Scofield analysis: double modal interchange and the use of multiple scale genres within a single musical event.

Example 7.4. Double modal interchange as set determinant

Analysis	$I^{\Delta 7}$	VI^{-7}	V^{7}	$I^{\Delta 7}$
Stated Harmony	$C^{\Delta 7}$	A^{-7}	G^{7}	$C^{\Delta 7}$
Chord/Scale: Common tone	C Ionian	C Ionian	C Ionian	C Ionian
Chord/Scale: Parallel	C Ionian	A Aeolian	G Mixolydian	C Ionian
Analysis	$I^{\Delta 7}$	[ ]	$V^{7\ \mathrm{sus}\flat 9}$	$I^{-\Delta 7}$
Reharmonization	$C^{\Delta 7}$	$A\flat^{-7\flat 5\ \natural 9}$	$G^{7\ \mathrm{sus}\flat 9}$	$C^{-\Delta 7}$
Chord/Scale: Common tone	C Ionian	C♭ real melodic minor	F real melodic minor	$\mathrm{Hex}_{(3,4)}$
Chord/Scale: Parallel	C Ionian	A♭ Locrian ♯2	G Dorian ♭2	$\mathrm{Hex}_{(3,4)}$

$A\flat^{-7\flat 5\ \natural 9}$ takes its root from C Aeolian and its quality from a differing scale genre. It is an example of double modal interchange. Keeping with the real melodic minor scale genre, the dominant chord is expressed as a sus chord with an altered 9th, a sonority previously observed in “The Beatles,” and a chord commonly employed to impose the real melodic minor sound over a dominant functioning harmony. The final chord, $C^{-\Delta 7}$, is inversionally related to a $C^{\Delta 7\sharp 5}$ at I_{11}: the hexatonic scale applies to both $\Delta 7\sharp 5$, and to $-\Delta 7$. We now have thee unique scale genres from which to define sets.

Diatonic and modal interchange chords can stand alone or be embellished with dominant action chords. In the latter case, diatonic and modal interchange chords sometimes act as target chords, the object of resolution for dominant action chords. The distinction between dominant action and target chords is crucial to understanding how jazz harmony works. Each category may take vastly different chord/scales. This is especially when the dominant action chord is altered, which it often is.

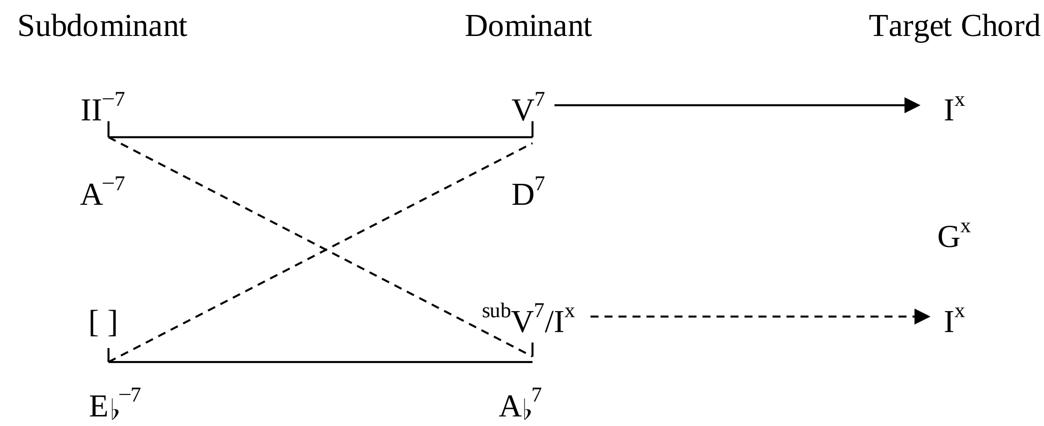
2.4. Dominant Action

Dominant action chords come in two common forms, the true dominant and the substitute dominant. The true dominant is a dominant seventh chord (less often a major triad) whose root lays perfect fifth above the target chord. The substitute dominant, shown as ${}^{sub}V^7/X$, is a dominant seventh chord (less often a major triad) whose root lays a half-step above the target chord *X*.[93] For any diatonic collection, there exists a single primary dominant (V^7), five unique secondary dominants $\{V^7/II^{-7}, V^7/III^{-7}, V^7/IV^{\Delta7}, V^7/V^7, V^7/VI^{-7}\}$, and six unique ${}^{sub}V^7/X$ harmonies $\{{}^{sub}V^7/I^{\Delta7}, {}^{sub}V7/\ II^{-7}, {}^{sub}V^7/III^{-7}, {}^{sub}V^7/IV^{\Delta7}, {}^{sub}V^7/V^7, {}^{sub}V^7/VI^{-7}\}$.

The II—V complex generalization holds that *any* dominant action chord may be preceded by its relative II^{-} (or $II^{-7\flat5}$), thus forming a II—V complex, and creating four possible dominant-action pathways to any target chord, shown in Example 8. In the following examples, target chords take the analytical expression I^x where variable *x* represents any chord quality other than minor seventh flat five, or fully diminished seventh. The choice of G as the tonic root is an arbitrary choice for illustrative purposes only. Some relative II^{-} harmonies hold dual functions, as (1) diatonic or modal interchange chords and as (2) the relative II^{-} chord to their dominant partners. B^{-7} in the key of G major is an example. B^{-7} functions as both III^{-7} and the relative II^{-7} of V^7/II^{-}. We use the notation of empty brackets, [ ], to represent relative II^{-} chords irreconcilable as diatonic or modal interchange chords, functioning only as the subdominant partner to the corresponding dominant action chord.

93 See, Nicole Biamonte, "Augmented-Sixth Chords vs. Tritone Substitutes." *Music Theory Online* 14, no.2 (2008), for further work on this relationship where the replacement of the cycle-five root motion with a descending minor second equates with Robert Morris's TTO transformation T_6MI (T_6M_7).

Example 8.1. II—V complex derivation



Example 8.2. II—V possibilities with a single target chord

1.	II^{-7} A^{-7}	V^{7} D^{7}	I^{x} G^{x}
2.	[ ] $E\flat^{-7}$	${}^{sub}V^{7}/I^{x}$ $A\flat^{7}$	I^{x} G^{x}
3.	II^{-7} A^{-7}	${}^{sub}V^{7}/I^{x}$ $A\flat^{7}$	I^{x} G^{x}
4.	[ ] $E\flat^{-7}$	V^{7} D^{7}	I^{x} G^{x}

This section closes with two analyses, both of which focus on dominant-action chords. The first one addresses Mick Goodrick's triad-over-bass-note reharmonization of "Rhythm Changes." The second looks at Thelonious Monk and Denzil Best's composition "Bemsha Swing." The analysis of "Bemsha Swing" introduces the dihedral group and geometric duality.

Analysis 3. Mick Goodrick's reharmonization of "Rhythm Changes"

Modal interchange and dominant-action chords in their triadic forms create an analytical problem, due to *Mehrdeutigkeit*. For example, consider D♭$^{\Delta}$ in the key of C major. Does D♭$^{\Delta}$ function as modal interchange, ♭II$^{\Delta 7}$, or as dominant-action, ^{sub}V/I? Analytical determination is open to an interpretation based on the composition's style and/or surrounding musical events. Given the harmonic nature of "Rhythm Changes," with its liberal inclusion of applied dominants and the extended dominant pattern occupying the bridge, the choice is made to treat such triads as dominant-action chords.[94]

Dominant-action explains the relationships held between the upper-structure triads and other musical forces, such as the stated harmony and bass line. One would suspect that the triads and bass notes work as a unit—this is not the case. The bass notes constitute a complete musical event separate from the upper structure triads, a point taken up in the analysis.

[94] An extended dominant pattern is a harmonic progression built from a sequential chain of dominant-action chords. Examples include the harmony of "Sweet Georgia Brown" and the bridge to "I've Got Rhythm."

Example 9. Mick Goodrick's reharmonization of “Rhythm Changes”

The following précis pertains to the bass notes.

mm. 1-4: The reharmonization opens on a four-bar dominant pedal releasing on

mm. 5-6: the first structural tonic, part of a minor-tonic consonant skip ($\hat{1} \rightarrow \flat\hat{3}$), followed by

mm. 6-8: cycle-five sequence presented as ascending pitch-interval 4, spanning $\hat{7}$ to $\hat{5}$.

mm. 9-12: A chromatic fifth-progression spanning ($\hat{1} \rightarrow \hat{5}$) ensues, followed by

mm. 13: a major-tonic consonant skip ($\hat{1} \rightarrow \natural\hat{3}$) correcting the initial minor inflected consonant skip in m.5.

mm.14-16: Cycle-five motion returns albeit in its inverted form, the descending pitch-interval 7, spanning $\hat{4} \rightarrow \hat{5}$. This final section is a IV—II complex where the descending cycle-five motion is an expansion of IV closing on $\hat{2}$—$\hat{5}$—$\hat{1}$.

The précis pertaining to triadic upper-structures focuses on the triad's dominant-action relationship to either: (1) an adjacent triad, (2) an adjacent bass note, or (3) adjacent stated harmony. To reduce visual clutter, extended dominant patterns are shown using eighth-note beams. As analytical symbols, the significance of a solid beam and a dotted beam hold to the same definitions as dominant-action arrows. Because all but the last three triads are major, the use of delta as the symbol for major is suspended in this analysis.

mm. 1-6: Triads relate to the stated harmony except for one deviation in m.3 where the triad relates to the subsequent bass note. A tri-tone exchange (D^{Δ}—$A\flat^{\Delta}$) precedes a single ^{sub}V that moves to a Janus chord that looks forward as a ^{sub}V to the stated harmony, and looks backward as a back-relating dominant of the previous bass note. The roots of the first twelve triads attain a twelve-tone row (10,1,9,11,0,4,6,2,8,7,5,3), B♭ = 0.

mm. 6-8: The last note of the row statement begins a series of three exchanges resembling chromatic voice exchanges between the triadic root and bass note.

mm. 8-10: Triads form the exchange (01), where {0 = ^{sub}/V, 1 = dominant}.

mm. 10-11: The first chord in each measure contains a bass note which holds a dominant relationship with both the subsequent triad and bass note. ^{sub}/V motion passes between triads to bass notes.

m.12: A tritone swap precedes a second Janus chord where the triad looks backward as a back-relating dominant to the preceding bass note and also looks forward as the ${}^{sub}V/I$, resolving into the tonic bass note in,

mm. 14-16: ${}^{sub}/V$ extended dominants close at the arrival of bass note $\hat{2}$. The minor triads in the final two triads provide harmonic extensions over the stated harmony, and are not part of the dominant-action scheme.

Analysis 4. "Bemsha Swing"

The composition "Bemsha Swing," co-written by Thelonious Monk and Denzil Best, uses dominant-action harmonies as its principal harmonic force. The structural harmonic background is reminiscent of a typical blues: a clearly stated initial tonic and the ubiquitous motion to the subdominant (in m.9). Secondary dominants and ${}^{sub}/V$ chords intermingle in a harmonic *spiel* that expands dominant-action usage beyond what has been discussed thus far. Primary dominant, secondary dominant and ${}^{sub}/V$ harmonies coexist in extended dominant patterns rife with deceptive resolutions that move into other dominant-action target chords. Take the progression A^7—$A\flat^7$—$D\flat^7$ in mm.1-2. In the home key of C major, these chords represent V^7/II^{-7}, ${}^{sub}V^7/V^7$, ${}^{sub}V^7/I^{\Delta}$. However, in "Bemsha Swing," A^7 acts as the ${}^{sub}V^7$ of ${}^{sub}V^7/\ V^7$ ($A\flat^7$), $A\flat^7$ acts as V^7 of ${}^{sub}V/I^{\Delta}$ ($D\flat^7$), and $D\flat^7$, the ${}^{sub}V^7/I^{\Delta}$ that finally resolves, as expected, into C^{Δ}, m.3. This type of dominant-action motion occurs throughout the composition. Example 10 is an annotated lead sheet containing a functional analysis.

Example 10. “Bemsha Swing”

Coda is for final melodic recapitulation

In “Bemsha Swing,” Monk utilizes dominant-action chords as extended embellishments of basic diatonic harmony, an underlying $I^{\Delta} \rightarrow IV^{\Delta} \rightarrow I^{\Delta}$. His choice of root motions, displayed as directed interval classes, reveal an additional structure, one that is permutational in nature. Let x = interval class, + = ascending, and − = descending. Example 11 displays the composition’s directed interval classes. Harmonic phrases, shown as h_n, are comprised of two parts, $h_n = y + z$. Phrase h_5, the coda, includes harmonies from m.11 to form a complete phrase. The final chord,

$A\flat^{\Delta 7}$, occurs only once, and acts as an appendage to the form proper. It causes the only directed i.c. 4 within the composition and is excluded from Example 11, we shall return to it.

Example 11. “Bemsha Swing,” root motion as directed interval classes

mm. phrase h_n	*y*				*z*				
1-5 h_1	C^{Δ}	A^7	$A\flat^7$	$D\flat^7$	C^{Δ}	$E\flat^7$	D^7	$D\flat^7$	(C^{Δ})
	-3	-1	+5	-1	+3	-1	-1	-1	
5-9 h_2	C^{Δ}	A^7	$A\flat^7$	$D\flat^7$	C^{Δ}	$B\flat^7$	$A\flat^7$	$G\flat^7$	(F^{Δ})
	-3	-1	+5	-1	-2	-2	-2	-1	
9-13 h_3	F^{Δ}	D^7	$D\flat^7$	$G\flat^7$	F^{Δ}	$A\flat^7$	G^7	$D\flat^7$	(C^{Δ})
	-3	-1	+5	-1	+3	-1	$\pm$6	-1	
13-16 h_4	C^{Δ}	A^7	$A\flat^7$	$D\flat^7$	C^{Δ}	$E\flat^7$	D^7	$D\flat^7$	(C^{Δ})
	-3	-1	+5	-1	+3	-1	-1	-1	
coda 11, 17-20 h_5	F^{Δ}	$A\flat^7$	G^7	$D\flat^7$	C^{Δ}	A^7	$A\flat^7$	$D\flat^7$	C^{Δ}
	-3	-1	$\pm$6	-1	-3	-1	+5	-1	

Let the set $\boldsymbol{A} := \{-1, -2, -3, +3, +5, \pm 6\}$, the set of all directed i.c.s in Example 11. Figure 5 contains the six elements of $\boldsymbol{A}$ placed on the vertices of a regular hexagon. The full symmetry group of the regular hexagon is isomorphic to the dihedral group on six elements, D_{12}.

Definition 13. Dihedral group

Dihedral groups can be generated by two classes of group actions, rotations about a point in a plane and a reflection about an axis of symmetry. It is notated as D_n, the dihedral group of order $2n$, and has the following group presentation:[95]

$$D_n := \langle r,h \mid r^n, h^2, (rh)^2 = i \rangle.$$

Dihedral groups describe the full symmetry group of a regular n-gon where $n \geq 3$, and as shall be seen later, the full symmetry group of certain Platonic solids. Rotations are the members of the cyclic subgroup C_n; the reflection is a member of a subset of involutions that takes every set element q to q'. If set members x and y lay on the line representing the axis of symmetry, x and y are stabilized.[96]

Define a group E as the full symmetry group of the regular hexagon acting on $\boldsymbol{A}$, $E := (\boldsymbol{A}, D_{12})$.

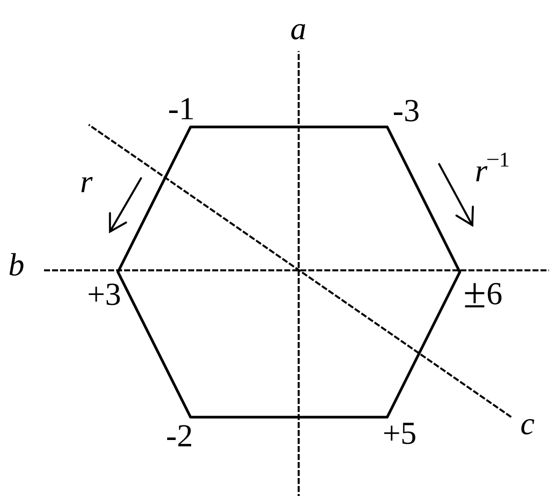


Figure 5. $E := (\boldsymbol{A}, D_{12})$

[95] The group presentation reads as follows, the first entry are the group generators, followed by | (such that), and concluding with relations. The group K is the group generated by r and h, such that, r^3, h^2, and $(rh)^2$ return the group identity element, i.
[96] For additional explanations of dihedral groups, see, Budden, 187-213; and Gallian 34-6.

Define $\boldsymbol{B}$:= {k = [-3,-1,+5,-1], p = [+3,-1,±6,-1], s = [-2,-2,-2,-1], t = [+3,-1,-1,-1]} as the set of permutations of $\boldsymbol{A}$ that are found in the composition. Elements of $\boldsymbol{B}$ derive from the group actions of E, shown below.

$$k := [\overbrace{-3,-1}^{r},\underbrace{+5,-1}_{a}] \qquad p := [\overbrace{+3,-1}^{r^{-1}},\underbrace{\pm 6,-1}_{b}] \qquad s := \underset{c}{[+3,-1]} \qquad t := \underset{b}{[-2,-1]}$$

Example 12 lists members of $\boldsymbol{B}$ according to phrase structure.

Example 12. Directed interval class permutations in phrases *hn*

mm. phrase h_n	Permutation	sub-phrase y	Permutation	sub-phrase z
1-5 h_1	k	[-3,-1,+5,-1]	t	[+3,-1,-1,-1]
5-9 h_2	k	[-3,-1,+5,-1]	s	[-2,-2,-2,-1]
9-13 h_3	k	[-3,-1,+5,-1]	p	[+3,-1,±6,-1]
13-19 h_4	k	[-3,-1,+5,-1]	t	[+3,-1,-1,-1]
coda 11, 17-20 h_5	p	[+3,-1,±6,-1]	k	[-3,-1,+5,-1]

Permutations k and p have a more complex structure than that of t or s. Permutations k and p both start with some form of i.c. 3 moving to directed i.c. −1, followed by an exchange involving i.c.−1. Thus, k and p contain two group actions, a rotation that maps the first element to the second element, followed by permutational mutation into the exchange. The cyclic members of k and p exhibit the inverse relation; they share orthogonal axes of reflection, a and b; t and s each contain two set elements, and therefore are exchanges belonging to a set of involutions in E.

Permutation k occurs in all y sub-phrase except h_5, the coda, where p relegates k to the final z sub-phrase, a reflection of h_3. To further investigate how phrase structure influences the reading of ***B***, label the elements of ***B*** as integers where $\{1= k, 2= p, 3 = s, 4 = t\}$ and define permutations according to phrase structure: $\{h_1 = (14), h_2 = (13), h_3 = (12), h_4 = (14), h_5 = (21)\}$. Every permutation in S_n, $n > 1$, is a product of exchanges.[97] Therefore, the product of the 2-cycles determined by phrase structure $(14)(13)(12) = (1234)$ agrees with r in C_4. The permutation in h_5 adds a reflection, thus creating D_8. Define the group $F := (\boldsymbol{B}, D_8)$. Two dihedral groups are at work here: E explains how the chords' roots are represented as directed interval classes and form the set ***B***, containing Monk's E permutations. The other copy of D_8, F, models the elements of ***B*** delineated by phrase structure.

Let us revisit the final sonority A♭$^{\Delta}$ in its triadic form. Define the group T as the transposition group acting on the set of twelve major triads. Redefine $s \in E$ (the reflection through axis c) as the function composition $(r^4 \cdot a)$.

$$\overbrace{r \cdot a}^{k} \cdot \overbrace{r^{-1} \cdot b}^{p} \cdot \overbrace{b}^{t} \cdot \overbrace{r^4 \cdot a}^{s}$$

$$(r \cdot r^{-1})a \cdot b \cdot b \cdot r^4 \cdot a \qquad \text{by involution}$$

$$a \cdot (b \cdot b) \cdot r^4 \cdot a \qquad \text{by involution}$$

$$(a \cdot a) \cdot r^4 \qquad \text{by involution}$$

$$r^4$$

With the involutions reduced out, the action r^4 remains.

[97] See, Gallian, 121-2.

Further analysis regarding the final sonority requires two additional definitions, conjugation and normal subgroup.

Definition 14. Conjugation

"Assume G is a group. Conjugation is an equivalence relation defined in G: two group members a and b are conjugate if there exists $g \in G$ such that $a = g^{-1}bg$, thereby partitioning the set G into disjoint equivalence classes of conjugate elements, known as conjugacy classes."[98] Conjugation determines the outcome of one group element under another group element. Accordingly, we write b^g, and say b under (the action of) g. This usage is opposed to the language that describes the multiplicative action bg, in which we say b followed by g, or simply b by g.

Conjugations of group elements form an automorphism class, known as inner automorphisms of $G \rightarrow G$. Budden explains,

> If x remains fixed, while y runs through all elements [members] of the group, i.e. we consider the elements xy_1x^{-1}, xy_2x^{-1}, xy_3x^{-1}, …, xy_nx^{-1}, then we can show that the correspondence $y_r \leftrightarrow xy_rx^{-1}$ constitutes an automorphism.[99]

The following example adapts Budden's discussion to show inner isomorphisms of D_6 using our notation. The example is then reinterpreted as D_6 acting on the set of the three unique octatonic collections, to model T/I sections and to illustrate conjugation musically. D_6 actions are: $\{i, r, r^{-1}, f(a), f(b), f(c)\}$. For a geometric representation of D_6, refer to Figure 2.

Example 13. Conjugation as inner automorphism

$$\left.\begin{aligned} rir^{-1} &= i \\ rrr^{-1} &= r \\ rr^{-1}r^{-1} &= r^{-1} \\ rar^{-1} &= c \\ rbr^{-1} &= a \\ rcr^{-1} &= b \end{aligned}\right\} \begin{aligned} i &\leftrightarrow i \\ r &\leftrightarrow r \\ r^{-1} &\leftrightarrow r^{-1} \\ a &\leftrightarrow c \\ b &\leftrightarrow a \\ c &\leftrightarrow b \end{aligned}$$

[98] Moore, 105.

[99] Budden, 152.

Assign the following *T/I* actions to the actions of D_6: (r = T_4, r^{-1} = T_8, a = I_4, b = I_0, c = I_2), let these actions act on the set of unique octatonic collections. Permutations are shown in cyclic notation and actions on the triangle.

$$i = T_0 = (Oct_{(0,1)})\ (Oct_{(1,2)})\ (Oct_{(2,3)}) = \begin{array}{ccc} & \mathrm{Oct}_{(0,1)} & \\ \mathrm{Oct}_{(1,2)} & & \mathrm{Oct}_{(2,3)} \end{array}$$

$$T_{1,4,7,10} = (Oct_{(0,1)},\ Oct_{(1,2)},\ Oct_{(2,3)}) = \begin{array}{ccc} & \mathrm{Oct}_{(2,3)} & \\ \mathrm{Oct}_{(0,1)} & & \mathrm{Oct}_{(1,2)} \end{array}$$

$$T_{2,5,8,11} = (Oct_{(01)},\ (Oct_{(2,3)},\ Oct_{(1,2)}) = \begin{array}{ccc} & \mathrm{Oct}_{(1,2)} & \\ \mathrm{Oct}_{(2,3)} & & \mathrm{Oct}_{(0,1)} \end{array}$$

$$a = I_{1,4,7,10} = (Oct_{(1,2)},\ Oct_{(2,3)}) = \begin{array}{ccc} & \mathrm{Oct}_{(0,1)} & \\ \mathrm{Oct}_{(2,3)} & & \mathrm{Oct}_{(1,2)} \end{array}$$

$$b = I_{0,3,6,9} = (Oct_{(0,1)},\ Oct_{(2,3)}) = \begin{array}{ccc} & \mathrm{Oct}_{(2,3)} & \\ \mathrm{Oct}_{(1,2)} & & \mathrm{Oct}_{(0,1)} \end{array}$$

$$c = I_{2,5,8,11} = (Oct_{(0,1)},\ Oct_{(1,2)}) = \begin{array}{ccc} & \mathrm{Oct}_{(1,2)} & \\ \mathrm{Oct}_{(0,1)} & & \mathrm{Oct}_{(2,3)} \end{array}$$

The reinterpretation of Budden's example gives examples of i^{T_y}, ${\mathrm{T}_x}^{\mathrm{T}_y}$ and ${\mathrm{I}_x}^{\mathrm{T}_y}$. To these, we include ${\mathrm{T}_x}^{\mathrm{I}_y}$ and ${\mathrm{I}_x}^{\mathrm{I}_y}$, shown in Table 1.

Table 1. *T*/*I* conjugation

T/I Conjugate	Action	Inner Automorphism	Formula
i^{T_y}	$\genfrac{}{}{0pt}{}{(0,1)}{(1,2)\ (2,3)} \xrightarrow{\mathrm{T}_4} \genfrac{}{}{0pt}{}{(2,3)}{(0,1)\ (1,2)} \xrightarrow{\mathrm{T}_0} \genfrac{}{}{0pt}{}{(2,3)}{(0,1)\ (1,2)} \xrightarrow{\mathrm{T}_8} \genfrac{}{}{0pt}{}{(0,1)}{(1,2)\ (2,3)}$	$rir^{-1} = i$	$i^{\mathrm{T}_y} = i$
${\mathrm{T}_4}^{\mathrm{T}_4}$	$\genfrac{}{}{0pt}{}{(0,1)}{(1,2)\ (2,3)} \xrightarrow{\mathrm{T}_4} \genfrac{}{}{0pt}{}{(2,3)}{(0,1)\ (1,2)} \xrightarrow{\mathrm{T}_4} \genfrac{}{}{0pt}{}{(1,2)}{(2,3)\ (0,1)} \xrightarrow{\mathrm{T}_8} \genfrac{}{}{0pt}{}{(2,3)}{(0,1)\ (1,2)}$	$rrr^{-1} = r$	$\mathrm{T}_x^{\mathrm{T}_y} = \mathrm{T}_x$
${\mathrm{T}_8}^{\mathrm{T}_4}$	$\genfrac{}{}{0pt}{}{(0,1)}{(1,2)\ (2,3)} \xrightarrow{\mathrm{T}_4} \genfrac{}{}{0pt}{}{(2,3)}{(0,1)\ (1,2)} \xrightarrow{\mathrm{T}_8} \genfrac{}{}{0pt}{}{(0,1)}{(1,2)\ (2,3)} \xrightarrow{\mathrm{T}_8} \genfrac{}{}{0pt}{}{(1,2)}{(2,3)\ (0,1)}$	$rr^{-1}r^{-1} = r^{-1}$	$\mathrm{T}_{x^{-1}}^{\mathrm{T}_y} = \mathrm{T}_{x^{-1}}$
${\mathrm{I}_1}^{\mathrm{T}_4}$	$\genfrac{}{}{0pt}{}{(0,1)}{(1,2)\ (2,3)} \xrightarrow{\mathrm{T}_4} \genfrac{}{}{0pt}{}{(2,3)}{(0,1)\ (1,2)} \xrightarrow{\mathrm{I}_1} \genfrac{}{}{0pt}{}{(2,3)}{(1,2)\ (0,1)} \xrightarrow{\mathrm{T}_8} \genfrac{}{}{0pt}{}{(1,2)}{(0,1)\ (2,3)}$	$rar^{-1} = c$	$\mathrm{I}_x^{\mathrm{T}_y} = \mathrm{I}_{x+2y}$
${\mathrm{I}_0}^{\mathrm{T}_4}$	$\genfrac{}{}{0pt}{}{(0,1)}{(1,2)\ (2,3)} \xrightarrow{\mathrm{T}_4} \genfrac{}{}{0pt}{}{(2,3)}{(0,1)\ (1,2)} \xrightarrow{\mathrm{I}_0} \genfrac{}{}{0pt}{}{(1,2)}{(0,1)\ (2,3)} \xrightarrow{\mathrm{T}_8} \genfrac{}{}{0pt}{}{(0,1)}{(2,3)\ (1,2)}$	$rbr^{-1} = a$	$\mathrm{I}_x^{\mathrm{T}_y} = \mathrm{I}_{x+2y}$
${\mathrm{I}_2}^{\mathrm{T}_4}$	$\genfrac{}{}{0pt}{}{(0,1)}{(1,2)\ (2,3)} \xrightarrow{\mathrm{T}_4} \genfrac{}{}{0pt}{}{(2,3)}{(0,1)\ (1,2)} \xrightarrow{\mathrm{I}_2} \genfrac{}{}{0pt}{}{(0,1)}{(2,3)\ (1,2)} \xrightarrow{\mathrm{T}_8} \genfrac{}{}{0pt}{}{(2,3)}{(1,2)\ (0,1)}$	$rcr^{-1} = b$	$\mathrm{I}_x^{\mathrm{T}_y} = \mathrm{I}_{x+2y}$
${\mathrm{T}_4}^{\mathrm{I}_0}$	$\genfrac{}{}{0pt}{}{(0,1)}{(1,2)\ (2,3)} \xrightarrow{\mathrm{I}_0} \genfrac{}{}{0pt}{}{(2,3)}{(1,2)\ (0,1)} \xrightarrow{\mathrm{T}_4} \genfrac{}{}{0pt}{}{(0,1)}{(2,3)\ (1,2)} \xrightarrow{\mathrm{I}_0} \genfrac{}{}{0pt}{}{(1,2)}{(2,3)\ (0,1)}$	$brb^{-1} = r^{-1}$	$\mathrm{T}_x^{\mathrm{I}_y} = \mathrm{T}_{x^{-1}}$
${\mathrm{I}_1}^{\mathrm{I}_2}$	$\genfrac{}{}{0pt}{}{(0,1)}{(1,2)\ (2,3)} \xrightarrow{\mathrm{I}_2} \genfrac{}{}{0pt}{}{(1,2)}{(0,1)\ (2,3)} \xrightarrow{\mathrm{I}_1} \genfrac{}{}{0pt}{}{(1,2)}{(2,3)\ (0,1)} \xrightarrow{\mathrm{I}_2} \genfrac{}{}{0pt}{}{(2,3)}{(1,2)\ (0,1)}$	$aca^{-1} = b$	$\mathrm{I}_x^{\mathrm{I}_y} = \mathrm{I}_{x^{-1}+2y}$

Definition 15. Normal subgroup

"A subgroup *H* of a group *G* is called a normal subgroup of *G* if $aH = Ha$ for all *a* in *G*. *H* is normal in *G* if and only if $xHx^{-1} \subseteq H$ for all *x* in *G*. It is written $H \lhd G$."[100]

Returning to the final chord in "Bemsha Swing," inscribe a regular hexagon within the regular dodecagon. Let the lines of the dodecagon represent the elements of the set ***D*** := the twelve major triads, and let the group *T* be the transposition group (C_{12}) acting on ***D***, $T := (\boldsymbol{D}, C_{12})$. Let the hexagonal vertices represent elements of the set ***F*** be the rotational permutations of *E*. Define the group *H* as C_6 acting on ***F***, $H := (\boldsymbol{F}, C_6)$, and put $H \lhd T$. Plot the point *q*, representing $r^4 \in H$; plot the line q' on *T* to show the directed i.c. −4 that maps $C^{\Delta} \mapsto A\flat^{\Delta}$. Therefore, we say *q* and q' share the same point, which lives in the geometric duality held between $H \lhd T$.

[100] Gallian, 172-3.

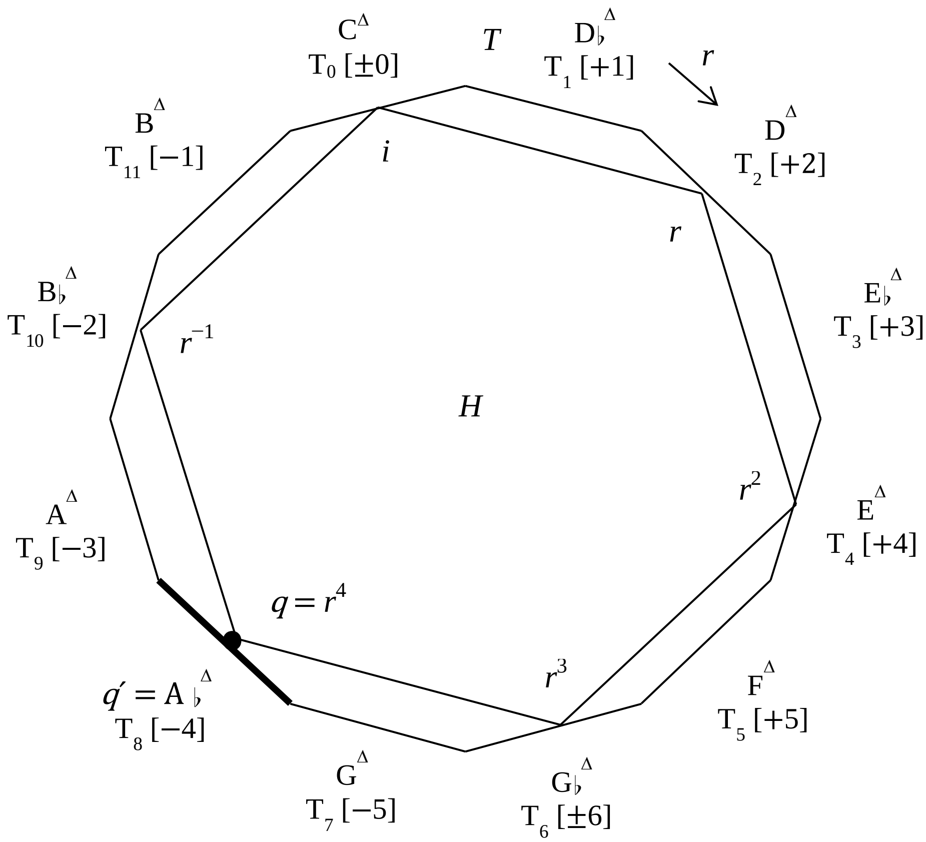


Figure 6. Geometric duality and "Bemsha Swing's" final sonority[101]

2.5. Tonic Systems

Tonic systems, written X^{T}, are harmonic units derived from symmetrical divisions of the octave.[102] Most often, the chordal quality contained within a tonic system is of constant-structure, meaning, each chord that is contained within the tonic system is of an invariant quality. For example, consider the major triads in Coltrane's "Giant Steps," or the $\mathrm{C}^{\Delta7\sharp5} \rightarrow \mathrm{E}\flat^{\Delta7\sharp5} \rightarrow \mathrm{A}^{\Delta7\sharp5}$

---

[101] This is a duality between the six-sided figure and the twelve-sided figure. The quotient of the twelve-sided figure is modulo the subgroup generated by $\langle T_6 \rangle$, $T/\langle T_6 \rangle$.

[102] Scale collections that divide the octave symmetrically serve as tonic system generators, e.g., octatonic, hexatonic, or nonatonic (ennatonic). These collections are also referred to as modes of limited transposition. See, Olivier Messiaen, *La technique de mon langage musical*, trans. John Satterfeld (Paris: Alphonse Leduc, 1956), 58-63; John Schuster-Craig, "An Eighth Mode of Limited Transposition," *The Music Review* 51, no. 4 (1990): 296-306; Jack Douthett and Peter Steinbach, "Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition," *Journal of Music Theory* 42, no.2 (1998): 241-63.

chords in"The Beatles." There is, however, no strict requirement to this practice—one may include non-invariant chord qualities as long as all harmonic pitch content contained within the tonic system corresponds to a symmetric scale.

The generating interval class defines the tonic system, whereby the tonic system descriptor *X* equals the order of their orbit. Therefore, tonic system (G^{Δ}, $E\flat^{\Delta}$, B^{Δ}) represents an orbit of order 3, in which three iterations of T_4 produce self-coincidence, a three-tonic system, written 3^{T}, and may be derived from either the hexatonic collection or the nonatonic collection.[103] An orbit of order 4 with four T_3 iterations effecting self-coincidence, (G^{Δ}, E^{Δ}, $D\flat^{\Delta}$, $B\flat^{\Delta}$), represents a four-tonic system, written 4^{T}. This is derived from the octatonic collection.[104] Harmonies contained within a tonic system, like diatonic and modal interchange chords, may act as target chords, the object of their dominant-action II—V complexes, shown below.

Example 14.1. Two tonic system [2^{T}]

$$
\begin{array}{cccccccccc}
 & [\ ] & ({}^{sub}V^{7}/I^{x}) & & & II^{-7} & V^{7} & \\
2^{T} & E\flat^{-7} & A\flat^{7} & & 2^{T} & A^{-7} & D^{7} & 2^{T} \\
 & |\_\_\_\_\_\_| & \searrow & & & |\_\_\_\_\_\_| & \searrow & \\
G^{x} & & & D\flat^{x} & & & & G^{x} \\
 & II^{-7} & (V^{7}) \nearrow & & [\ ] & {}^{sub}V^{7}/I^{x} \nearrow & & \\
 & A^{-7} & D^{7} & & E\flat^{-7} & A\flat^{7} & & \\
 & |\_\_\_\_\_\_\_| & & & |\_\_\_\_\_\_\_| & & &
\end{array}
$$

[103] The nonatonic collection is the union of two copies of the hexatonic collections: $\text{nonatonic}_{(0,1,2)}$ = (0,1,2,4,5,6,8,9,10) = $\text{Hex}_{(0,1)} \cup \text{Hex}_{(1,2)}$. In this document, three-tonic systems are treated as part of the hexatonic collection exclusively.

[104] The whole tone collection is an anomaly. Although examples of six-tonic systems do exist, for example, {G^{Δ}, F^{Δ}, $E\flat^{\Delta}$, $D\flat^{\Delta}$, B^{Δ}, A^{Δ}}, only the root motion holds to the generative symmetric scale; the union of all triadic pitches attains the chromatic aggregate.

Example 14.2. Three tonic system [3^T]

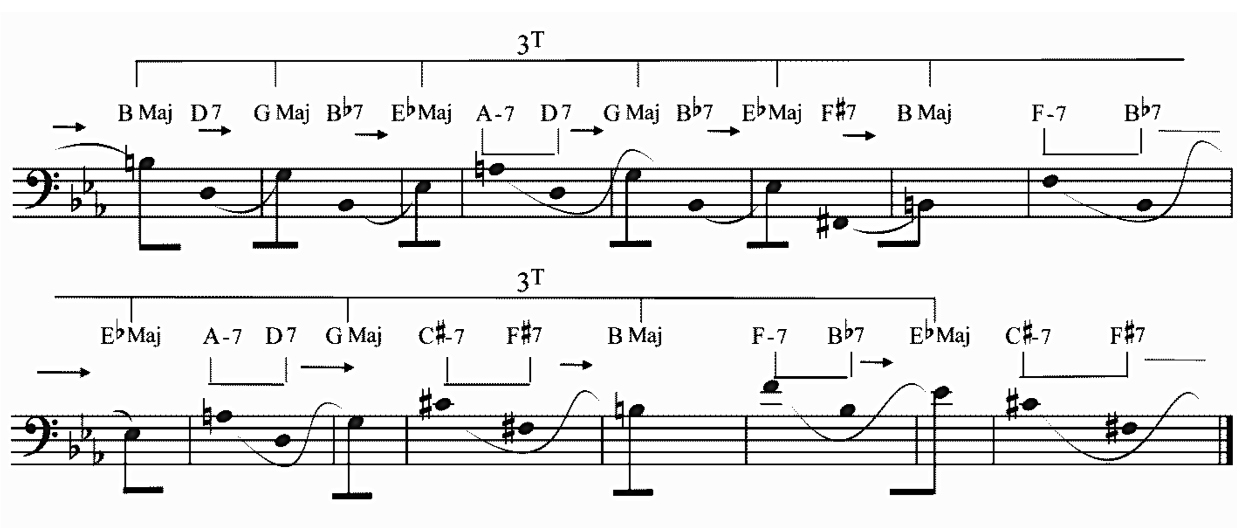
	$\flat VII^{-7}$	$(^{sub}V^7/II^{-7})$		$\sharp IV^{-7\flat5}$	(V^7/III^{-7})		II^{-7}	V^7	
3^T	F^{-7}	$B\flat^7 \searrow$	3^T	$C\sharp^{-7\flat5}$	$F\sharp^7 \searrow$	3^T	A^{-7}	$D^7 \searrow$	3^T
G^x			$E\flat^x$			B^x			G^x
	III^{-7}	$(V^7/II^{-7}) \nearrow$		I^{-7}	$(^{sub}V^7/III^{-7}) \nearrow$		[ ]	$^{sub}V^7/I^x \nearrow$	
	B^{-7}	E^7		G^{-7}	C^7		$E\flat^{-7}$	$A\flat^7$	

Example 14.3. Four tonic system [4^T]

	$VII^{-7\flat5}$	(V^7/VI^{-7})		[ ]	$^{sub}V^7/I^x$		IV^{-7}	$(^{sub}V^7/VI^{-7})$		II^{-7}	V^7	
4^T	$F\sharp^{-7\flat5}$	$B^7 \searrow$	4^T	$E\flat^{-7}$	$A\flat^7 \searrow$	4^T	C^{-7}	$F^7 \searrow$	4^T	A^{-7}	$D^7 \searrow$	4^T
G^x			E^x			$D\flat^x$			$B\flat^x$			G^x
	IV^{-7}	$(^{sub}V^7/VI^{-7}) \nearrow$		II^{-7}	$V^7 \nearrow$		$VII^{-7\flat5}$	$(V^7/VI^{-7}) \nearrow$		[ ]	$^{sub}V^7/I^x \nearrow$	
	C^{-7}	F^7		A^{-7}	D^7		$F\sharp^{-7\flat5}$	B^7		$E\flat^{-7}$	$A\flat^7$	

Example 14.2 is the 3^T system contained Coltrane's "Giant Steps," derived from $Hex_{(1,2)}$. The other harmonies are dominant-action chords.

Example 15. "Giant Steps," graphic analysis



The term tonic system carries with it the inference of a multi-tonic construction, a term sometimes used to describe this type of harmonic unit. This association is somewhat fallacious, however, as it suggests that each harmonic component is cognitively processed as a discrete tonic that is established through modulation. If one instead considers the requirements of modulation—one being, the requisite confirmation of the new key—the idea of tonic systems as a series of modulations falls short of meeting such a definition. As such, tonic systems can exist within compositions that are based in a key without invoking a strict definition of modulation (which would imply that other keys have replaced the home key). To illustrate, consider the motion $I^7 \rightarrow IV^7$ in a $B\flat$ blues. We want to superimpose harmonies that create a sense of motion into the IV^7 chord. An introductory option is to draw upon dominant-action harmonies.

Example 16. Harmonic superimposition, $^{sub}V^7/IV^7$

Chord/Scale		E Lydian dominant	
Analysis		[ ] $^{sub}V^7/IV^7$	
Superimposition		B^{-7} E^7	
Chord/Scale	$B\flat$ Mixolydian		$E\flat$ Mixolydian
Analysis	I^7		IV^7
Stated Harmony	$B\flat^7$		$E\flat^7$

We could superimpose a 3^T system from $Hex_{(2,3)}$ to provide motion into IV^7, see Example 17; or combine the two techniques by leading into the 3^T system's first chord with a triadic representation of its ^{sub}V, C^Δ. Initially, C^Δ sounds as an upper-structure triad to $B\flat^7$, implying the harmonic tensions 9, ♯11, 13, alluding to a $B\flat$ Lydian dominant chord/scale choice. Only upon resolution to B^{Δ}, the first 3^{T} chord, do we recognize C^{Δ} as a 3^{T} preparation rather than an extension of $B\flat^{7}$.

Example 17. Harmonic superimposition, 3^{T}

Chord/Scale III		B♭ Lydian dominant		$\mathrm{Hex}_{(2,3)}$	
Chord/Scale II		C Lydian	B Lydian	G Lydian	E♭ Lydian Dominant
Chord/Scale I		C Ionian	B Ionian	G Ionian	E♭ Mixolydian
Analysis		$^{Sub}V/B^{\Delta}$		3^{T}	
Superimposition		C^{Δ}	B^{Δ}	G^{Δ}	$E\flat^{\Delta}$
Chord/Scale	B♭ Mixolydian				E♭ Mixolydian
Analysis	I^{7}				IV^{7}
Stated Harmony	$B\flat^{7}$				$E\flat^{7}$

The technique in the previous example shows that tonic systems are not strictly multi-tonic modulatory constructions that threaten to destroy an established key; rather, they coexist with harmonies that have clearly defined functional implications without destroying those implications. Moreover, tonic systems hold the possibility of strengthening the aural perception of the functional implications held by the harmonies they embellish.

Coltrane exploited this quality in his contrafactual writing and improvisations.[105] Coltrane's contrafacts, not only replace the existing melody, as in a typical contrafact, but the harmonic foreground as well. Interpolated chromatic mediant relationships, based upon tonic

[105] Contrafacts are compositions where the composer takes an existing composition, and replaces the existing melody with a new one. This was a popular practice in the be-bop era to thwart copyright laws.

systems, embellish the original composition’s structural events. Walt Weiskopf and Ramon Ricker cite numerous examples, compiled in Example 18.[106]

Example 18. Coltrane contrafacts

Original Composition	Composer	Coltrane Contrafact
“Tune-Up”	Eddie Vinson (often attributed to Miles Davis)	“Countdown”
“But Not for Me”	George Gershwin	“Fifth House”
“Confirmation”	Charlie Parker	“26-2”
‘How High the Moon”	Morgan Lewis	“Satellite”
“I Can’t Get Started”	Vernon Duke	“Exocita”

This harmonic concept emancipates the triad from the relationship held with the stated harmony of a standard tune. In Example 19, tonic systems act as reharmonizations in excerpts from two standard tunes, the relatively static harmony contained in the first eight bars of Gene dePaul’s “I’ll Remember April” and the extended dominant pattern in the bridge of “Rhythm Changes.” Reharmonizations also serve as a basis for chord/scale choices, as was done in the B♭ blues example prior. In the following examples, the stated harmony occupies the bottom row, bold type highlights members of the tonic system, and arrows show dominant-action resolution.

Example 19.1. 3^T reharmonization, first eight bars of “I’ll Remember April,” containing triads from $Hex_{(2,3)}$

<table>
<tr><td>Measure</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td></tr>
<tr><td>Reharmonization</td><td colspan="4">$\mathbf{G^{\Delta}}$ $B\flat^7 \rightarrow \mathbf{E\flat^{\Delta}}$ $F\sharp^7 \rightarrow \mathbf{B^{\Delta}}$ $D^7 \rightarrow \mathbf{G^{\Delta}}$</td><td colspan="4">$\mathbf{G^{-}}$ $B\flat^7 \rightarrow \mathbf{E\flat^{-}}$ $F\sharp^7 \rightarrow \mathbf{B^{-}}$ $D^7 \rightarrow \mathbf{G^{-}}$</td><td></td></tr>
<tr><td>Stated Harmony</td><td colspan="4">G^{Δ}</td><td colspan="4">G^{-}</td><td>$A^{-7\flat5}$</td></tr>
</table>

[106] Walt Weiskopf and Ramon Ricker, *Giant Steps: A Player’s Guide to Coltrane’s Harmony for All Instrumentalists* (New Albany: Jamey Aebersold Jazz, 1991).

Example 19.2. 3^T reharmonization, first eight bars of "I'll Remember April," containing triads from $Hex_{(2,3)}$ and $Hex_{(1,2)}$

Measure	1	2	3	4	5	6	7	8	9
Reharmonization	$\mathbf{G}^{\Delta}$	F^{-7} $B\flat^7$	$\mathbf{E\flat}^{\Delta}$ $F\sharp^7$	$\mathbf{B}^{\Delta}$ D^7	$\mathbf{C\sharp}^{\Delta}$ $B\flat^7$	$\mathbf{A}^{\Delta}$ $G\flat^7$	$\mathbf{F}^{\Delta}$ $A\flat^7/C$	$\mathbf{D\flat}^{\Delta}$ $B\flat^7$	
Stated Harmony	G^{Δ}				G^-				$A^{-7\flat5}$

Example 19.3. 3^T reharmonization of the bridge to "I've Got Rhythm," containing triads from $Hex_{(1,2)}$ and $Hex_{(11,0)}$

Measure	1	2	3	4	5	6	7	8	9
Reharmonization	D^7 F^7	$\mathbf{B\flat}^{\Delta}$ $D\flat^7$	$\mathbf{G\flat}^{\Delta}$ A^7	$\mathbf{D}^{\Delta}$ G^7	C^7 $E\flat^7$	$\mathbf{A\flat}^{\Delta}$ B^7	$\mathbf{E}^{\Delta}$ G^7	$\mathbf{C}^{\Delta}$ F^7	
Stated Harmony	D^7		G^7		C^7		F^7		$B\flat^{\Delta}$

In the example above, the C^7—F^7 reharmonization section is a T_{10} image of the D^7—G^7 reharmonization.

Example 19.4. 3^T reharmonization of the bridge to "I've got Rhythm," containing triads from $Hex_{(3,4)}$ and $Hex_{(1,2)}$

Measure	1	2	3	4	5	6	7	8	9
Reharmonization	D^7 G^7	$\mathbf{C}^{\Delta}$ $E\flat^7$	$\mathbf{A\flat}^{\Delta}$ B^7	$\mathbf{E}^{\Delta}$ G^7	C^7 B^7	$\mathbf{B\flat}^{\Delta}$ G^7	$\mathbf{G\flat}^{\Delta}$ $E\flat^7$	$\mathbf{D}^{\Delta}$ F^7	
Stated Harmony	D^7		G^7		C^7		F^7		$B\flat^{\Delta}$

In Example 19.4, triads over D^7—G^7 derive from $Hex_{(3,4)}$ while triads over C7—F7 derive from $Hex_{(1,2)}$.

Through the tonic system, we obtain an aurally identifiable harmonic device that influenced contrafact composition, reharmonization, and chord/scale choices. Coltrane defined a harmonic set generated by symmetrical divisions of the octave, and used this set as a stylistic hallmark, so much so that jazz musicians use the colloquial term "Coltrane changes" to describe this musical technique.[107] While 3^T systems are what jazz musicians are describing with this colloquial term, Coltrane did not limit himself to 3^T systems. In the next analysis, we see Coltrane's use of the 4^T system.

Analysis 5. "The Father, and the Son, and the Holy Ghost"

Coltrane recorded the *Meditations* suite in 1965. In the suite's first movement, "The Father, and the Son, and the Holy Ghost," Coltrane presents a melodic gesture built upon an ascending major triad arpeggio that contain an added passing tone, equating to the triad's $\hat{2}$. David Liebman discusses the symmetry between the suite movements in terms of the keys that the individual movements suggest, the form of the suite as a whole, and the use of rhythm and musical intensity among the various movements.[108] The symmetry addressed in this analysis involves pitch content, specifically, 4^T systems.

Define the set $\boldsymbol{X} := \{1 = \mathrm{Oct}_{(0,1)}, \mathrm{Oct}_{(2,3)}, \mathrm{Oct}(_{1,2})\}$ and the group $K := (\boldsymbol{X}, D_6)$.[109] Further, define the following sets as three copies of the set of major triads obtained from the octatonic collections:

$$\boldsymbol{T}^{[1]} := (\mathrm{Oct}_{(0,1)}) = \{1 = \mathrm{G}\flat^{\Delta}, 2 = \mathrm{E}\flat^{\Delta}, 3 = \mathrm{C}^{\Delta}, 4 = \mathrm{A}^{\Delta}\}),$$

$$\boldsymbol{T}^{[2]} := (\mathrm{Oct}_{(1,2)}) = \{1 = \mathrm{G}^{\Delta}, 2 = \mathrm{E}^{\Delta}, 3 = \mathrm{D}\flat^{\Delta}, 4 = \mathrm{B}\flat^{\Delta}\}),$$

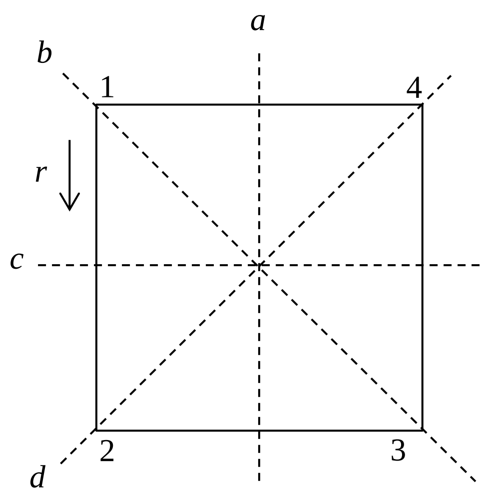
$$\boldsymbol{T}^{[3]} := (\mathrm{Oct}_{(2.3)}) = \{1 = \mathrm{A}\flat^{\Delta}, 2 = \mathrm{F}^{\Delta}, 3 = \mathrm{D}^{\Delta}, 4 = \mathrm{B}^{\Delta}\}).$$

Define the group $Z := D_8^3$. We note that D_8 acts individually on each of the three copies of $\boldsymbol{T}^{[1\ldots3]}$. Figure 7 is a geometric representation of D_8.

[107] While jazz musicians commonly refer to this technique as "Coltrane changes," however, this harmonic technique existed much earlier, traceable to at least C.P.E. Bach. See Matthew Bribitzer-Stull, "The A♭—C—E Complex: The Origin and Function of Chromatic Third Collections in Nineteenth-Century Music." *Music Theory Spectrum* 28 no.2 (2006): 167-190.

[108] David Liebman, "John Coltrane's Meditations Suite: A Study in Symmetry," *Annual Review of Jazz Studies* 8 (1996): 167-80. The transcription of "The Father, the Son, and the Holy Ghost" is on page 170 of Liebman's article. a reproduction is contained in Appendix B.3.

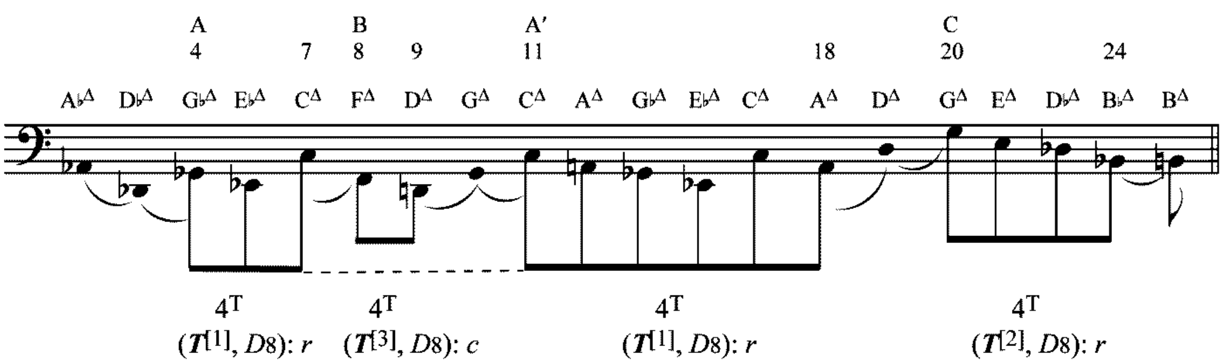
[109] For a geometric representation of D_6, see Figure 2.



C_4
$i := (1)(2)(3)(4)$
$r := (1234)$
$r^2 := (13)(24)$
$r^{-1} := (1432)$
$D_8 := \langle C_4, f_x \in f\,(a \ldots d)\rangle$
$a := (14)(23)$
$b := (24)$
$c := (12)(34)$
$d := (13)$

Figure 7. Symmetries of the square $\cong D_8$

Example 20. “The Father, and the Son, and the Holy Ghost”



The harmonic content includes triads from three 4^T systems and their corresponding dominant-action chords. 4^T iterations delineate the music into four distinct parts: A = mm. 4-7, B = mm. 8-9, A′ = mm. 11-18, C = mm. 20-24. Triadic inversional positions support this reading. The majority of triads are in second position except: m.4, the first triad in section A; m. 11, the first triad in section A′; and m. 20, the first triad in section C. Dominant-action chords fill mm. 1-3, m. 10, and m. 19.

An extended dominant progression $A\flat^{\Delta}$—$D\flat^{\Delta}$ ushers in $G\flat^{\Delta}$, the initial triad of section A, which is modeled by using group action $(\boldsymbol{T}^{[1]}, D_8)$: r. The action on $\boldsymbol{T}^{[1]}$ ceases on C^{Δ}. It becomes a dominant-action chord to F^{Δ}, the initial triad of section B. F^{Δ} progresses to D^{Δ}, which begins a second extended dominant pattern. The reinterpretation of D^{Δ} as an extended dominant chord ends section B, thus defining a group action of order 2. We consider this action to be an exchange in $(\boldsymbol{T}^{[2]}, D_8)$: c. The second extended dominant pattern returns C^{Δ}, the abandoned triad in section A. $(\boldsymbol{T}^{[1]}, D_8)$: r continues through section A′, ultimately completing two $(\boldsymbol{T}^{[1]}, D_8)$: r cycles. A′ ends on A^{Δ}, which becomes the first chord of a final extended dominant pattern, leading to G^{Δ} and the arrival of section C. This final motion is modeled with the permutation $(\boldsymbol{T}_3, D_8)$: r.

"The Father, The Son and The Holy Ghost" is another example of a direct product of two groups, $Z \times K$, where Z describes the triads within sectional boundaries and K explains the group action that generates the sections defined by the octatonic collections. Section A = $\mathrm{Oct}_{(0,1)} = \boldsymbol{X}_1$, section B = $\mathrm{Oct}_{(2,3)} = \boldsymbol{X}_2$, section A′ = $\mathrm{Oct}_{(0,1)} = \boldsymbol{X}_1$, section C = $\mathrm{Oct}_{(1,2)} = \boldsymbol{X}_3$. Therefore, K: b,c represents K: (1,2),(1,3) the group actions in K that map the octatonic collections. The analysis tells us that Coltrane's use of 4^{T} system material takes the form $Z \times K \cong D_8^3 \times D_6$.

The next analysis considers 3^{T} and 4^{T} systems working in concert in Willis Delony's and William Grimes's arrangement of Richard Rodgers's and Lorenz Hart's "Have You Met Miss.

Jones." The analysis also expands the definition of the direct product to include actions of the familiar musical transposition group T_n on major triads.

Analysis 6. "Have You Met Miss. Jones"

"Have You Met Miss. Jones" is an early example of a tonic system, carrying a publication date of 1937. The song in in key of F major and has an AABA form. The formal B section, containing the tonic system is of interest here.

Example 21. 3^T system in "Have You Met Miss. Jones," formal section B

$IV^{\Delta 7}$	[ ]	$V^7/G\flat^{\Delta 7}$		[ ]	$V^7/D^{\Delta 7}$		[ ]	$V^7/G\flat^{\Delta 7}$		II^{-7}	V^7
$B\flat^{\Delta 7}$	$A\flat^{-7}$	$D\flat^7$	$G\flat^{\Delta 7}$	E^{-7}	A^7	$D^{\Delta 7}$	$A\flat^{-7}$	$D\flat^7$	$G\flat^{\Delta 7}$	G^{-7}	C^7

3^T

The novel aspect of the arrangement is a consistent T_3 modulation for each subsequent chorus during the solo section. This means that after four choruses, the T_3 transpositional level returns the harmony to the home key. After passing through a series of 3^T systems, and when taken as a set union, this system attains all twelve major triads. To analyze the arrangement, we involve the group theoretical concept of an external direct product, which creates a new group from certain subgroups. This is the type of direct product we have used thus far.[110] Our goal is to explain the creation of a group structure isomorphic to C_{12} that describes the above arrangement and a method musicians might use to organize the music cognitively. Figure 8 is a geometric modeling of the twelve major triads partitioned into 3^T systems. Define the set $\boldsymbol{X}$:= { all major triads}, and define the group $Y := (\boldsymbol{X}, C_{12})$. Y is isomorphic to the transposition group T_n acting on $\boldsymbol{X}$.

[110] There are two types of direct products. An internal product is a decomposition of a group into certain subgroups and the external direct product is the creation of a new group from certain subgroups. A definition follows.

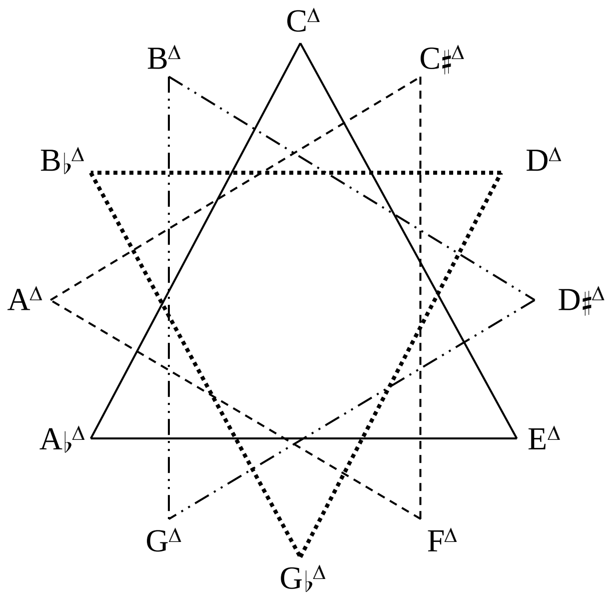


Figure 8. Group *Y*

While the four unique tonic systems are clearly apparent in Figure 19, this reading weighs heavily on the side of the stated harmony. The spirit of the arrangement is subordinate, as the T_3 component between choruses is not addressed. To do so, one would need to show the 3^T aspect of the arrangement. In Figure 8, this requires a rotation of three points to the right or left on the triangle modeling $\{B\flat^{\Delta}, G\flat^{\Delta}, D^{\Delta}\}$ to arrive at the subsequent chorus's pitch level. This interpretation begs the questions, is this truly the structure of the arrangement and is this how the musicians approach playing over the harmony?

Let us first consider the composers' contribution to the arrangement. Define the set $\boldsymbol{A} := (B\flat^{\Delta}, G\flat^{\Delta}, D^{\Delta})$, and the group $E := (\boldsymbol{A}, C_3)$. When playing the tune the typical fashion, each chorus remains at an invariant pitch level, represented as the set $\boldsymbol{A}$ at T_0, or E: i. Whereas this mapping seems a rather rudimentary concept in set theory, in group theory, elements that map to the identity element in a homomorphism form an important normal subgroup called the *kernel* of a homomorphism. By studying the kernel, we gain greater insight into the arrangement.

Definition 16. Kernel of a homomorphism

Let G and G' be two groups, the operation in each case being indicated as multiplication. A homomorphism of G into G' is the mapping $f : G \to G'$ such that $(ab)f = (af)(bf)$, for arbitrary $a, b \in G$. The kernel of a homomorphism of a group is the subset of G that maps onto the identity element (i) of G', written Ker(f),and defined as

$$\{x \in G \mid f(x) = i \in G'\}.$$ [111]

Let ***X*** remain as the set of all 12 triads and let the group Y be the transposition group acting on ***X***; let the transposition group be shown as the function f; and the group Y' as the four unique 3^{T} systems in chromatic space that are acted upon by f. As shown in Example 22.1, three members of f, $\{T_0, T_4, T_8\}$, map Y to the identity of Y'. Therefore, Ker(f) in $Y \xrightarrow{f} Y'$ is $\{T_0, T_4, T_8\}$.

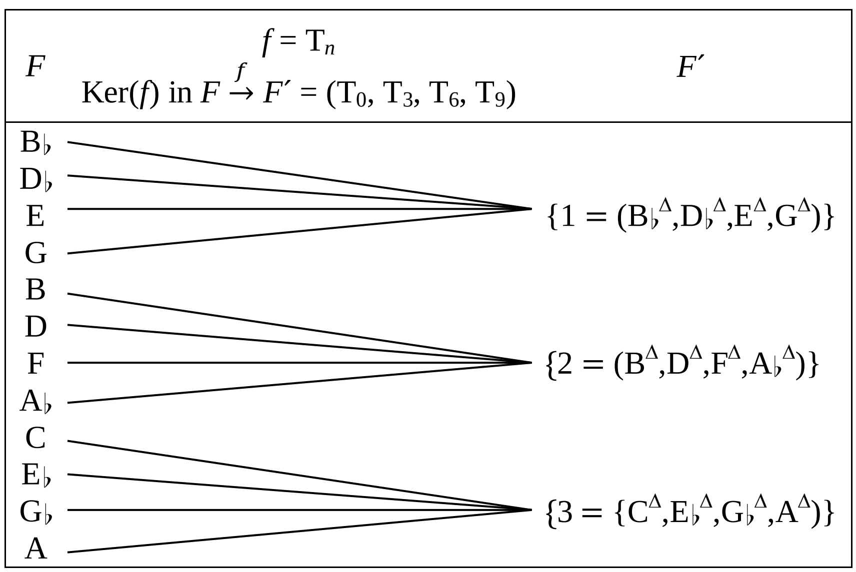
Example 22.1. Ker(f) in $Y \xrightarrow{f} Y'$

Y	$f = T_n$ Ker(f) in $E \xrightarrow{f} E' = (T_0, T_4, T_8)$.	Y'
$B\flat^{\Delta}$		
$G\flat^{\Delta}$		$\{1 = (B\flat^{\Delta}, G\flat^{\Delta}, D^{\Delta})\}$
D^{Δ}		
B^{Δ}		
G^{Δ}		$\{2 = (B^{\Delta}, G^{\Delta}, E\flat^{\Delta})\}$
$E\flat^{\Delta}$		
C^{Δ}		
$A\flat^{\Delta}$		$\{3 = (C^{\Delta}, A\flat^{\Delta}, E^{\Delta})\}$
E^{Δ}		
$D\flat^{\Delta}$		
F^{Δ}		$\{4 = (D\flat^{\Delta}, F^{\Delta}, A^{\Delta})\}$
A^{Δ}		

[111] Gallian, 194-200; Moore, 108-10.

The arrangers' contribution is the T_3 cycle that brings the 3^T system's pitch level to self-coincidence, which is a member of the kernel of the homomorphism $F \xrightarrow{f} F'$. Define the set ***B*** as all twelve pitch classes partitioned into subsets generated by T_3. ***B*** := ({B♭,D♭,E,G}), ({B,D,F,A}), {C,E♭,G♭,A}). The T_3 transpositional levels used in the arrangement are the orbit restriction ***C*** := (B♭,D♭,E,G) in ***B***.[112] Elements in ***C*** represent the first chords for each chorus. For example, the first chorus starts on $B\flat^{\Delta}$, then unfolds a 3^T system; the second chorus starts on $D\flat^{\Delta}$ then unfolds a 3^T system, the third chorus starts on E^{Δ} and unfolds a 3^T system, and so on. Therefore, the arrangement produces four copies of ***A***, one copy for each element in ***C***. The T_3 transposition level between choruses belongs to Ker(f) in $F \xrightarrow{f} F' = (T_0,T_3,T_6,T_9)$.

Example 22.2. Ker(f) in $F \xrightarrow{f} F'$



[112] When considering an orbit restriction, we look at the action of a group on a subset of its orbits (which may be one orbit). The orbit restriction may have a different permutation representation than the group itself.

In terms of group structure, the arrangers have introduced a group defined as $D := (\boldsymbol{C}, C_4)$, the transpositional level between choruses. It has an action on the composers' group defined as $E := (\boldsymbol{A}, C_3)$, the original 3^{T} system. The direct product of $E \times D$ produces the group $Y := (\boldsymbol{X}, C_{12})$.

Definition 17. Direct product

> A group Φ is the direct product of its subgroups $\Psi_1, \Psi_2, \ldots, \Psi_n$, and we write $\Phi := \Psi_1 \times \Psi_2 \times \ldots \times \Psi_n$, if either of the following sets of conditions is satisfied:
> (A) (1) The subgroups $\Psi_1, \Psi_2, \ldots, \Psi_n$ are normal in Φ.
> (2) $\Psi_1, \Psi_2, \ldots, \Psi_n = [\Psi_1, \Psi_2, \ldots, \Psi_n] = \Phi$.
> (3) $\Psi_g \cap (\Psi_1 \ldots \Psi_{g-1}\, \Psi_{g+1} \ldots \Psi_n) = i$, g = 1, 2, … n.
> (B) (1) $h_g h_j = h_j h_g$, for arbitrary $h_g \in \Psi_G$ and $h_j \in \Psi_J$, $g \neq j$, $g, j = 1,2, \ldots,$ n.
> (2) Each $k \in \Phi$ has a unique representation as a product $k := h_1 h_2 \ldots h_n$, where $hg \in \Psi_g$,
> $g := 1, 2, \ldots, n$.[113]

Holding to the notation in the above definition, let $\Phi := C_{12}$, so that for $a \in \Phi$, $a^{12} = i$. The elements $\{i, a^4, a^8\}$ (derived from the original harmony) comprise a (multiplicative) cyclic subgroup C_3, denoted Ψ_B. The elements $\{i, a^3, a^6, a^9\}$ (from the arrangement) make up a (multiplicative) cyclic subgroup C_4, denoted Ψ_D. The elements of Ψ_B and Ψ_D commute as they are cyclic subgroups of the cyclic group Φ. Each element of Φ has a unique representation in the form db,

$d \in \Psi_D$ and $b \in \Psi_B$: $i \cdot i = \mathrm{T}_0$, $d^9 \cdot b^4 = \mathrm{T}_1$, $d^6 \cdot b^8 = \mathrm{T}_2$, $d^3 \cdot b^i = \mathrm{T}_3$, $d^i \cdot b^4 = \mathrm{T}_4$, $d^9 \cdot b^8 = \mathrm{T}_5$, $d^6 \cdot b^i = \mathrm{T}_6$, $d^3 \cdot b^4 = \mathrm{T}_7$, $d^i \cdot b^8 = \mathrm{T}_8$, $d^9 \cdot b^i = \mathrm{T}_9$, $d^6 \cdot b^4 = \mathrm{T}_{10}$, $d^3 \cdot b^8 = \mathrm{T}_{11}$. As this agrees with the conditions stated in (B), the definition of the direct product $\Phi \cong \Psi_B \times \Psi_D$ proves the assertion $Y \cong E \times D$.

The original question posed at the beginning of the analysis remains. Is the group Y, in Figure 8, the best representation of how the arrangement works. Is it indicative of how musicians

[113] Adapted from Moore, 112-17.

cognitively organize the music? Let us consider another possibility by using a direct product to generate actions on $\boldsymbol{X}$. Define the group $Y' := (\boldsymbol{X}, C_{12}) \cong E^4 \times D$, modeled geometrically in Figure 9.

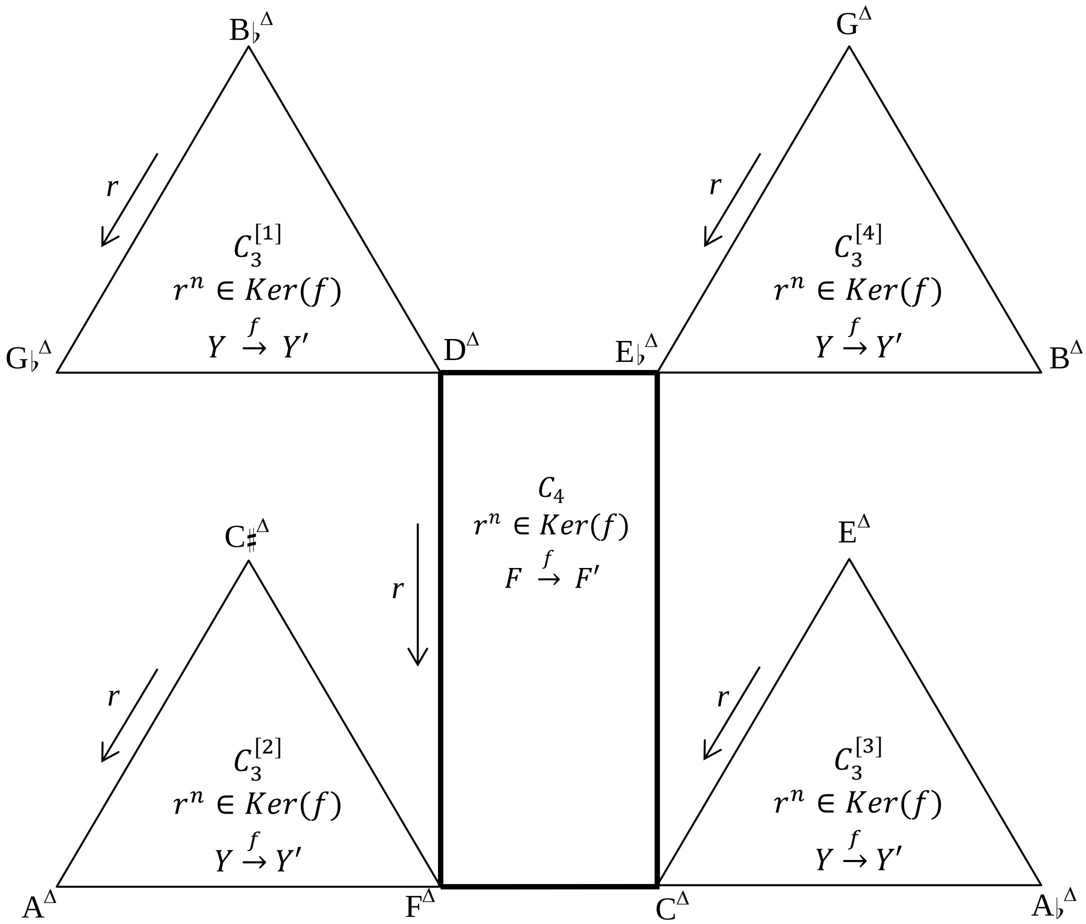


Figure 9. $Y' \cong E^4 \times D$

Compared to the initial modeling of Y in Figure 8, the direct product $C_3^4 \times C_4$ better explains the arrangement's structure and the mental organizational scheme musicians use to navigate the arrangement. Regarding chord/scale implications, the set union involving triads in each C_3^n copy attains a hexatonic collection.

That is, a possible chord/scale relationship for all the triads in any particular C_3^n *region* is the parent hexatonic: $C_3^{[1]}$ = $\text{Hex}_{(1,2)}$; $C_3^{[2]}$ = $\text{Hex}_{(0,1)}$; $C_3^{[3]}$ = $\text{Hex}_{(3,4)}$; $C_3^{[4]}$ = $\text{Hex}_{(2,3)}$. Organizing chord/scales as hexatonic collections in a *hexatonic region* provides a sonority other than the diatonic (Ionian or Lydian), and frees the player to accentuate additional music devices, or to treat dominant-action harmonies in a similar manner. We shall see the latter point addressed in a subsequent section of this study.

2.6. Chord/Scale Relationships

The chord/scale relationship explains a dual relationship between harmony and a set (scale) from which additional musical material can generate. For example, one can generate harmonic material from a scale, as is the case with modal harmony. On the other hand, one can also extract a scale from a harmony. There are numerous scale possibilities that color the harmony's sonority to varying degrees.[114] This subsection discusses several chord/scale relationship theories and provides an overview of methods that speak specifically to triadic usage.

2.6.1. The Aebersold/Baker Chord/Scale Method

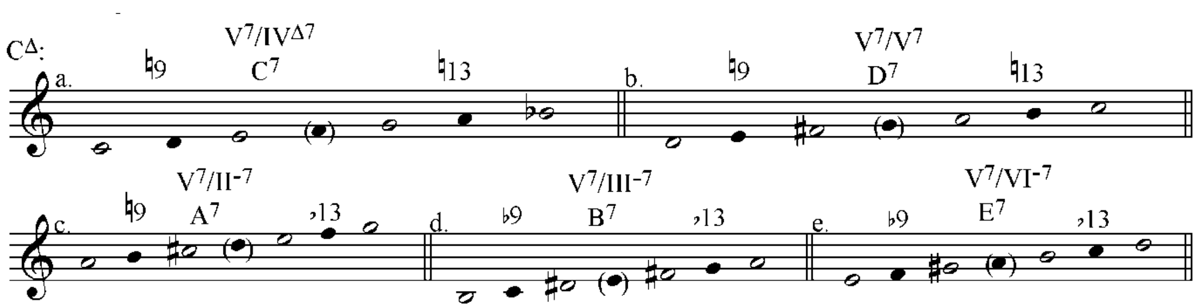
In his *Jazz Handbook,* Aebersold provides a *Scale Syllabus* that lists scales with their harmonic partners.[115] This system predicates scale choice entirely on chordal quality. Chord quality is but one possible chord/scale determinant, and restricting the criteria to chord quality leaves two questions unaddressed: the inclusion of harmonic function as chord/scale criteria, and the organization of multiple scale possibilities over a single harmony.

[114] Jazz musicians refer to this aural phenomenon as being either *inside* or *outside* the harmony's sonority.
[115] Aebersold, *Jazz Handbook*, 14. See Appendix C for a complete listing of Aebersold's chord/scale relationships.

2.6.2. Nettle's and Graf's *Chord/Scale Theory*

The Chord/Scale Theory utilizes chord function, not merely the chord quality, as a determining factor. Diatonic harmony takes its usual modal presentation, e.g., in a major key, II^{-7} takes Dorian, V^{7} takes Mixolydian, etc. As the chromatic content of a chord increases, so does the chromatic content of the scales. Secondary dominants, for instance, carry specific harmonic tensions when conceptualized within a key. As such, the method for secondary dominant chord/scale construction takes into account the chord's quality and the key in which it is functioning. The method is as follows: consider the seventh chord as a scale; fill in the missing pitches from the composition's key or the key of the moment (as in local tonicizations). With this method, we adhere to the proper harmonic extensions for each secondary dominant as they exist within a tonal framework. Example 23 lists the five secondary dominants. The primary dominant assumes Mixolydian.

Example 23. Secondary dominant chord/scale relationships[116]



A convenient way to organize secondary dominant chord/scales is to see them as the fifth mode of a scale built on the root of the intended chord of resolution. As to Example 23: (a) F Ionian (b) G Ionian (c) D real melodic minor, (harmonic minor is also common) (d) E harmonic minor (e) A harmonic minor.

---

[116] Graff, Nettles, 43.

The chord/scale choices for other functional harmonies are rather straightforward:

- Non-diatonic root dominant seventh chords (^{sub}V/*x* and dominant seventh quality modal interchange chords) take Lydian ♭7.
- Non-diatonic root minor seventh chords take Dorian
- Non-diatonic root major seventh chords take Lydian
- Non-diatonic root minor seventh flat five: (1) ♭II$^{-7\flat5}$, ♭VI$^{-7\flat5}$ take Locrian (2) ♭III$^{-7\flat5}$, ♭VII$^{-7\flat5}$ take Locrian ♯2; (3) ♯IV$^{-7\flat5}$ takes Locrian or Locrian ♯2, depending on context.
- Major seventh chords in tonic systems take Ionian or Lydian.
- Minor seventh chords in tonic systems take Aeolian or Dorian.

2.6.3. George Russell's *Lydian Chromatic Concept of Tonal Organization*

George Russell's *Lydian Chromatic Concept of Tonal Organization*, originally published in 1953, is an example of an early theory of chord/scale relationships. In it, Russell addresses the possibility of multiple scale choices of a single harmony by defining a set of scale collections he calls *principal scales*, all of which are available over a single harmony, see Example 24. The *parent scale* is Lydian or a modified form of Lydian. This scale best suits the chord quality and its available tensions and is chosen from among the *principal scales*. The choice of a parent scale imposes a hierarchy on the six additional scales, which offer varying degrees of chromaticism over the harmony. When taken as a set union, the seven principal scales attain the chromatic aggregate, hence the name "Lydian Chromatic."[117]

[117] Russell, 13.

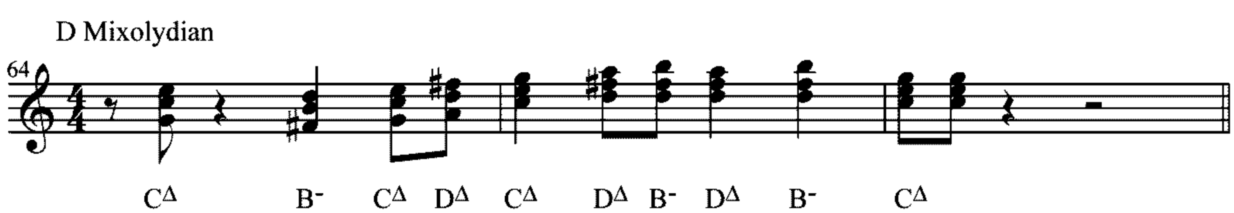
Example 24. Lydian Chromatic principal scales

Russell's Nomenclature	**Scale**	**Equivalency**
Lydian	$\hat{1},\hat{2},\hat{3},\sharp\hat{4},\hat{5},\hat{6},\hat{7}$	Fourth Mode , usual Diatonic
Lydian Augmented	$\hat{1},\hat{2},\hat{3},\sharp\hat{4},\sharp\hat{5},\hat{6},\hat{7}$	Third Mode, Real Melodic Minor
Lydian Diminished	$\hat{1},\hat{2},\flat\hat{3},\sharp\hat{4},\hat{5},\hat{6},\hat{7}$	Fourth Mode, Harmonic Major
Lydian Flat Seventh	$\hat{1},\hat{2},\hat{3},\sharp\hat{4},\hat{5},\hat{6},\flat\hat{7}$	Fourth Mode, Real Melodic Minor
Auxiliary Augmented	$\hat{1},\hat{2},\hat{3},\sharp\hat{4},\sharp\hat{5},\ \hat{7}$	Whole Tone
Auxiliary Diminished	$\hat{1},\hat{2},\flat\hat{3},\hat{4},\flat\hat{5},\flat\hat{6},{}^{o}\hat{7},\natural\hat{7}$	Whole/Half Diminished (Octatonic)
Auxiliary Diminished Blues	$\hat{1},\flat\hat{2},\flat\hat{3},\natural\hat{3},\sharp\hat{4},\hat{5},\hat{6},\flat\hat{7}$	Half/Whole Diminished (Octatonic)

For Russell, the parent scale determination of C Lydian for $C^{\Delta 7\sharp 11}$ is obvious. Other harmonic qualities are not so obvious. Take for example $E\flat^{7}$. Russell states that D♭ Lydian is the parent scale (D♭ Lydian = E♭ Mixolydian, both are rotations of the 4♭ diatonic); therefore, principal scales obtain from D♭ Lydian. How does Russell approach a complex harmony such as $E\flat^{7\sharp 5,\flat 9}$? In this case, the parent scale is a Lydian augmented scale built on the third of the $E\flat^{7}$ chord, G Lydian augmented, which is a member of E real melodic minor (equating to an E♭ diminished whole-tone). Therefore, any member of the G Lydian Chromatic—the principal scales that agree with G = $\hat{1}$—can be used over $E\flat^{7\sharp 5,\flat 9}$.[118]

Example 25 illustrates at the use of the Lydian Chromatic approach in an improvised solo; a chordal gesture from Pat Martino's solo on "Mardi Gras."[119]

Example 25. Pat Martino excerpt, "Mardi Gras"



---

[118] Russell, 79.
[119] Transcription by Jörg Heuser, *Pat Martino*: "*Joyous Lake*" (Mainz: BbArking Munckin Music, 2005), 13.

The harmony in the excerpt is D Mixolydian. Martino plays three unique chords defining the set $\boldsymbol{F} := \{1 = C^{\Delta}, 2 = D^{\Delta}, 3 = B^{-}\}$, $\boldsymbol{F} \subset \boldsymbol{D}_{(1\sharp)}$. C^{Δ} occupies the opening and closing structural downbeats (a delayed attack rhythmically embellishes the initial structural downbeat). The other chords, D^{Δ} and B^{-}, are characteristic chords in C Lydian. Martino's triadic gesture extends from the superimposition of C Lydian over D Mixolydian; C Lydian is the Lydian Chromatic Parent Scale for an unaltered D^{7} chord.

In terms of group structure, Martino uses the set of three involutions generated by $(\boldsymbol{F}, D_6)$: $a = (23)$, $b = (12)$, $c = (13)$.

$$\underbrace{C^{\Delta}\quad B^{-}}_{c}\quad \underbrace{C^{\Delta}\quad D^{\Delta}\quad C^{\Delta}}_{b}\quad \underbrace{D^{\Delta}\quad B^{-}\quad D^{\Delta}\quad B^{-}}_{a}\quad C^{\Delta}$$

As *Mehrdeutigkeit* describes an inherent ambiguity that functional harmonies possess, *Uneindeutigkeit* (not unique(ness)) describes the ambiguity inherent in chord/scale choice.[120] There is no *unique*, or single choice; a number of possibilities exist. For the jazz musician, *Uneindeutigkeit* is a tool used to convey musical affect by manipulating the levels of chromaticism in the musical domain. Russell's approach provides a manner in which organize *Uneindeutigkeit*.

2.7. Triad Specific Methods

We close the section on chord/scale relationships by looking at three improvisation methods that feature triads derived from the chord/scale relationship as their salient musical force. Pertinent mathematical properties held by the methods are also explained. This subsection

[120] This term is provided by the author.

also includes an analysis of a composition that shares the triadic set definition with one of the improvisation methods.

2.7.1. Gary Campbell's *Triad Pairs*

Gary Campbell's triadic method considers a six-note collection, generated by the set union of two discrete triad pairs, providing—according to Campbell—a more "concise sonority" than a seven-note scale. Campbell states,

> The structure and tensile strength of triads give the melodic line an independent internal logic…the stand alone sound is oftentimes enough to make a strong, effective melodic statement regardless of how it is (or isn't) relating to the harmony.[121]

Campbell draws upon seven scales to derive the triad pairs: (1) diatonic; (2) harmonic minor; (3) real melodic minor; (4) double harmonic [0,1,4,5,7,8,11] = (C,D♭,E,F,G,A♭,B); (5) harmonic major; and two symmetric scales, (6) hexatonic and (7) octatonic. To negate common tones, triads within a pair relate by either T_1, T_2, T_{10}, or T_{11}. Therefore, we say the set intersection of triads *j* and *k*, where *j* and *k* are members of the superset (scale) ***S*** contains no common tones,

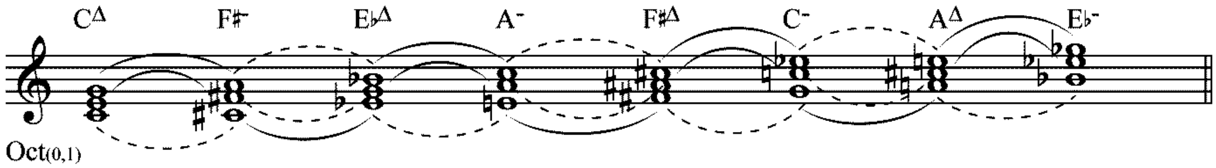
$$\{j,k \in \boldsymbol{S} \mid j \cap K = \emptyset\}.$$

Campbell introduces a novel approach to determine viable choices for the octatonic collection, which Campbell calls "auxiliary diminished," a term borrowed from Russell. Rather than being conjoined with its step-wise adjacency, the triad pair in octatonic space relies on the voice leading scheme displayed in Example 26, where two voices move by i.c. 2 (shown with a normal slur) and one voice moves by i.c.1 (shown with a broken slur). The voice leading scheme moves triadic elements by T_6 followed by T_9, and reverses parity ($\Delta \leftrightarrow -$).

---

[121] Campbell, 3.

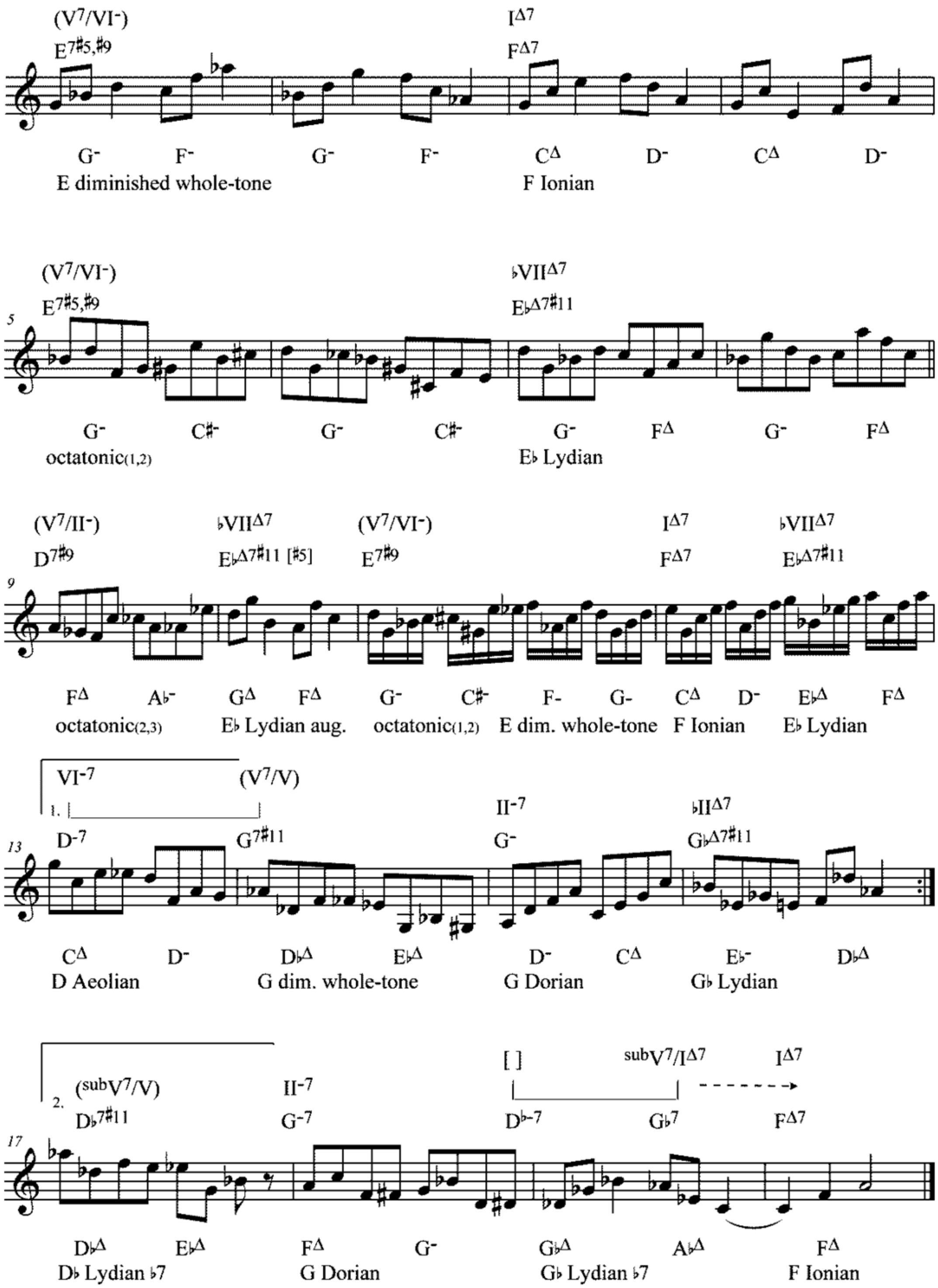
Example 26. Campbell's octatonic triad pairs



Since adjacent triads share no common-tones, and two adjacent triads may form a triad pair, this includes the "wrapping around" of E♭$^{-}$ and its adjoining with C$^{\Delta}$.

Example 27 is an etude based upon Campbell's technique over the harmony of Wayne Shorter's composition "E.S.P." The chord/scale choices follow Nettles and Graf by adhering to the chords' function in the key of F major, or by considering the stated tensions. In Campbell's text, he allows for the addition of a single pitch to the triad. The added pitch acts as either a passing tone or neighbor tone to a triad member. It can belong to the prescribed chord/scale, or it can be a chromatically inflected tone. This additional pitch helps to conform the triad's presentation rhythmically in duple meter.

Example 27. Triad pairs over the harmony to "E.S.P."



2.7.2. George Garzone's *Triadic Chromatic Approach*

In his Triadic Chromatic Approach, Garzone explains a theory of improvisation that borrows from "the twelve-tone row."[122] Chord/scale relationships are suspended and replaced with random triads that are coupled by step motion in pitch-class space.[123] The inversional position of adjacent triads cannot be invariant; a root-position triad cannot follow a root-position triad, a first-inversion triad cannot follow a first-inversion triad, and a second-inversion triad cannot follow a second-inversion triad. Otherwise, the triad's inversional positions are also random.

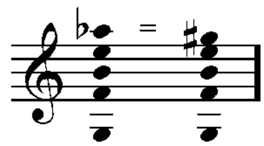
In his explanation, Garzone mentions the possibility of using dissonant triads, but he only offers examples that use consonant triads.[124] Since we are defining sets containing consonant triads, let us limit the following discussion to them. Garzone is describing a group structure on the set of consonant triads $\boldsymbol{E}$: = {consonant triads}. Let D_{48} act on $\boldsymbol{E}$, forming the group $M := (\boldsymbol{E}, D_{48})$. This group covers the actions among the triads. We must also consider an action on the inversional positions, of which there are three, defined as set $\boldsymbol{J}$, $\boldsymbol{J} := \{\frac{5}{3}, \frac{6}{3}, \frac{6}{4}\}$. Garzone allows for inversional position pairs, where two elements of $\boldsymbol{J}$ exchange. We may use D_6 to model $\boldsymbol{J}$; however, we cannot invoke the identity element of $\boldsymbol{J}$. The notation for the set of non-identity elements in a group X is indicated as $X^{\#}$. Therefore, define a subset of group $R := (\boldsymbol{J}, D_6^{\#})$. Garzone's approach uses this subset of a group structure (minus the identity element of $\boldsymbol{J}$). The larger group that he is describing is $G := M \times R$, the direct product of the groups M and R. Garzone's has defined a group, the product of $|M| = 48$ and $|R| = 5$: $|M||R| = 240$.

[122] Garzone, time code 1:00.
[123] The coupling may be an ascending or descending half-step.
[124] Discerning the inversion of an augmented triad would prove problematic.

2.7.3. Larry Carlton's Chord-Over-Chord Approach

In a 2003 interview, Larry Carlton explained his approach to melodic improvisation based on a chord-over-chord approach. The initial idea came from Carlton's enharmonic reinterpretation of an altered G dominant seventh chord.[125]

$G^{7\flat 9,13}$:



Instead of realizing the chord as a single entity, Carlton chooses a two-part composite, consisting of the basic seventh chord and a triad formed by the chord's 3rd (B) and the extensions ♭9 (A♭) and 13 (E). The resulting structure consists of an E^{Δ} triad lying above a G dominant seventh chord, hence Carlton's term "chord-over-chord."[126] He ultimately realized that three additional upper-structure triads, $B\flat^{\Delta}$, $D\flat^{\Delta}$, and G^{Δ} were also available. What Carlton describes is a triadic partitioning (restricted to major triads) of the $\mathrm{Oct}_{(1,2)}$ collection.

Throughout Carlton's description of his method, he tells us *what* triads to use; however, *how* to use the triads remains an open topic. To answer that question, and to understand Carlton's chord-over-chord approach in group theoretical terms, define the set $\boldsymbol{N} := \{1 = G^{\Delta}, 2 = B\flat^{\Delta}, 3 = D\flat^{\Delta}, 4 = E^{\Delta}\}$. We could consider the full symmetric group S_4 on these four elements, which yields 24 unique permutations. S_4 may be manageable for the musician. However, as the size of the set increases, the order of S_n quickly becomes unmanageable

---

[125] The class of an altered chord classifies that altered tensions in a more concrete description. The codification of altered chords remains nebulous in jazz practice and in the literature. Here, the class is ♭9,13, suggesting octatonic, therefore, ♯11 is assumed within this class of altered dominants.

[126] Larry Carlton, "Money Notes," *Guitar Player*, February 2003. Steve Masakowski gives this material as triads taken from one of the octatonic's two fully diminished seventh chords. See, Steve Masakowski, "Major symmetry, Diminished Treasure" *Guitar Player*, July, 1996.

musically, i.e., $|S_4| = 24$, $|S_5| = 120$, $|S_6| = 720$, $|S_7| = 5040$, and $|S_8|$, the full symmetric group of the 4 major and 4 minor triads contained within the octatonic collection , equals 40320. Therefore, to expand Carlton's set definition to include all consonant triads contained in the octatonic collection, applying only the information Carlton provides, the musician operates within a S_8 environment, wherein movement from triad to triad relies on aleatoric logic or strict aural guidance. While there are legitimate musical examples of aleatoric logic and aural guidance, a system of methodical organization of manageable set sizes offer much to the musician. In the next section of this document, Group Actions, we consider subgroups of S_n as a way to removing ourselves from the S_n world. Bur first let us consider a composition that uses Carlton's set $\boldsymbol{N}$ as a primary harmonic force.

Analysis 7. "Hotel Vamp"

In "Hotel Vamp," Steve Swallow utilizes the four major triads of the three unique octatonic collections. In fact, there are no functional harmonies anywhere in the ninety-six-measure, through-composed form. Aspects of the composition addressed in the following analysis include the following. (1) The four major triads holding T_3 relationships between each adjacent triad (the set $\boldsymbol{N}$ plus two additional copies of $\boldsymbol{N}$) contained in each four-measure segment of the tune. We label these segments as μ_nSegment, written more concisely as μ_nSeg, where n represents a specific μSeg of the 24 μSegs in the composition. The set union of any μSeg attains a unique octatonic collection, where

$$\boldsymbol{N}^{[3]} := \{ A\flat^{\Delta}, B^{\Delta}, D^{\Delta}, F^{\Delta}\} \in \mathrm{Oct}_{(2,3)},$$

$$\boldsymbol{N}^{[2]} := \{D\flat^{\Delta}, E^{\Delta}, G^{\Delta}, B\flat^{\Delta}\} \in \mathrm{Oct}_{(1,2)},$$

$$\boldsymbol{N}^{[1]} := \{A^{\Delta}, C^{\Delta}, E\flat^{\Delta}, G\flat^{\Delta}\} \in \mathrm{Oct}_{(0,1).}$$

This system is analogous to the four unique 4^T systems. (2) Three μSegs form a μBlock, concisely written as μ_nBk, that spans twelve measures. The composition contains eight μBks, and each μBk presents the three unique octatonic collections that are created by the μSeg set unions. (3) The triadic permutations contained in μSeg that manifest within μBk boundaries are cycles of order 2 and of order 3, generated by the semidirect product $C_4^3 \rtimes C_3$. We shall see that the permutations defined by this semidirect product belong to a group called the alternating group and that the products of alternating group members are significant to the analysis. (4) Permutations of the relationships held between the melody and harmony also form $C_4^3 \rtimes C_3$ and the resulting permutations also belong to the alternating group, and the products of their members hold analytical significance. Appendix B.4 contains an annotated lead sheet.

Example 28. “Hotel Vamp,” analytical synopsis

Triads																								
μ_nBk	1			2			3			4			5			6			7			8		
$\mu_{(n)}$Seg (mm)	1 (1-4)	2 (5-8)	3 (9-12	4 (13-16)	5 (17-20)	6 (21-24)	7 (25-28)	8 (29-32)	9 (33-36)	10 (37-40)	11 (41-44)	12 (45-48)	13 (49-52)	14 (53-56)	15 (57-60)	16 (61-64)	17 (65-68)	18 (69-72)	19 (73-76)	20 (77-80)	21 (81-84)	22 (85-88)	23 (89-92)	24 (93-96)
Oct$_{(x,y)}$	2,3	1,2	0,1	2,3	1,2	0,1	2,3	1,2	0,1	2,3	1,2	0,1	2,3	1,2	0,1	2,3	1,2	0,1	2,3	1,2	0,1	2,3	1,2	0,1
$J := (\boldsymbol{R}, C_3)$	r			r			r			r			r			r			r			r		
P $(\boldsymbol{N}^{[3]}, C_4)$	i			r^{-1}			i			i			i			r^2			i			r^{-1}		
P $(\boldsymbol{N}^{[2]}, C_4)$		i			r^2			r^{-1}			r^{-1}			r			r^{-1}			i			r^{-1}	
P $(\boldsymbol{N}^{[1]}, C_4)$			i			r^{-1}			r^{-1}			r^{-1}			r^{-1}			r			r			i
$W := P \rtimes J \in A_4$	$i = i$			$(43) = \pi_2$			$(14) = \pi_4$						$(124) = \gamma^{-1}$			$(342) = \alpha^{-1}$			$(124) = \gamma^{-1}$					
													$\pi_2\ \alpha^{-1} = \gamma^{-1}$			$\pi_2\ \gamma^{-1} = \alpha^{-1}$			$\pi_2\ \alpha^{-1} = \gamma^{-1}$					
Melody																								
$L := (\boldsymbol{F}, C_4)$	i	r^2	r^{-1}	r^2	r^{-1}	r	i	r	r^2	r^{-1}	i	r	i	r^{-1}	r^2	r	i	r^{-1}	i	r^2	i	r^2	i	r^2
$Z := B \rtimes J \in A_4$	$(134) = \beta$			$(342) = \alpha^{-1}$			$(123) = \delta$			$(412) = \gamma^{-1}$			$(143) = \beta^{-1}$			$(214) = \gamma$			$(13) = \pi_3$					
	$\alpha^{-1}\ \gamma^{-1} = \beta$			$\alpha^{-1} i = \alpha^{-1}$			$\gamma^{-1}\ \alpha^{-1} = \delta$			$\gamma^{-1} i = \gamma^{-1}$			$\pi_4\ \alpha^{-1} = \beta^{-1}$			$(\gamma^{-1})^2 = \gamma$								
hyper-μBk	$\beta \cdot \alpha^{-1} = \pi_3$						$\delta \cdot \gamma^{-1} = \pi_3$						$\beta^{-1} \cdot \gamma = \pi_3$											
hyper-μBk $(xy(z)) \in W$	$\alpha^{-1}\ \gamma^{-1}\ \alpha^{-1} = \pi_3$						$\gamma^{-1}\ \alpha^{-1}\ \gamma^{-1} = \pi_3$						$\pi_4\ \pi_2 = \pi_3$											

We begin with the analysis of µSeg material featuring three copies of $\boldsymbol{N}$. $\boldsymbol{N}$ is of degree 4. As such, the rotational symmetry of the square, isomorphic to the cyclic group on four elements, C_4 , serves as an appropriate group structure to model µSegs. Triadic assignments for $\boldsymbol{N}^{[x]}$ and the identity element (i) for each octatonic collection are listed below,

$(\boldsymbol{N}^{[1]}, C_4)$: i := {1 = A$^\Delta$, 2 = C$^\Delta$, 3 = E♭$^\Delta$, 3 = G♭$^\Delta$}, {1 ∪ 2∪ 3∪ 4} := Oct$_{(0,1)}$,

$(\boldsymbol{N}^{[2]}, C_4)$: i := {1 = D♭$^\Delta$, 2= E$^\Delta$, 3 = G$^\Delta$, 4 = B♭$^\Delta$},{1 ∪ 2∪ 3∪ 4} := Oct$_{(1,2)}$,

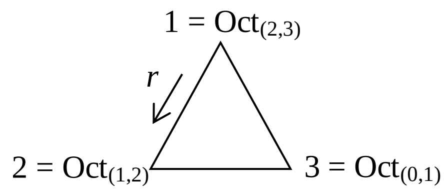
$(\boldsymbol{N}^{[3]}, C_4)$: i := {1 = A♭$^\Delta$, 2 = B$^\Delta$, 3 = D$^\Delta$, 4 = F$^\Delta$}, {1 ∪ 2∪ 3∪ 4} := Oct$_{(2,3)}$.

Define the group $P := (\boldsymbol{N}^{[1\ldots3]}, C_4^3)$, the µSeg group.

Example 29. µSeg group P

Octatonic	$(\boldsymbol{N}^{[x]}, C_4^3)$	P
(2,3)	$(\boldsymbol{N}^{[3]}, C_4^3)$	i := (1)(2)(3)(4) = (A♭$^\Delta$)(B$^\Delta$)(D$^\Delta$)(F$^\Delta$) r := (1234) = (A♭$^\Delta$,B$^\Delta$,D$^\Delta$,F$^\Delta$) r^2 := (13)(24) = (A♭$^\Delta$,D$^\Delta$)(B$^\Delta$,F$^\Delta$) r^{-1} := (1432) = (A♭$^\Delta$,F$^\Delta$,D$^\Delta$,B$^\Delta$)
(1,2)	$(\boldsymbol{N}^{[2]}, C_4^3)$	i := (1)(2)(3)(4) = (D♭$^\Delta$)(E$^\Delta$)(G$^\Delta$)(B♭$^\Delta$) r := (1234) = (D♭$^\Delta$,E$^\Delta$,G$^\Delta$,B♭$^\Delta$) r^2 := (13)(24) = (D♭$^\Delta$,G$^\Delta$)(E$^\Delta$,B♭$^\Delta$) r^{-1} := (1432) = (D♭$^\Delta$,B♭$^\Delta$,G$^\Delta$,E$^\Delta$)
(0,1)	$(\boldsymbol{N}^{[1]}, C_4^3)$	i := (1)(2)(3)(4) = (A$^\Delta$)(C$^\Delta$)(E♭$^\Delta$)(G♭$^\Delta$) r := (1234) = (A$^\Delta$,C$^\Delta$,E♭$^\Delta$,G♭$^\Delta$) r^2 := (13)(24) = (A$^\Delta$,E♭$^\Delta$)(C$^\Delta$,G♭$^\Delta$) r^{-1} := (1432) = (A$^\Delta$,G♭$^\Delta$,E♭$^\Delta$,C$^\Delta$)

To facilitate µBk analysis, we need to determine the mapping of the unique octatonic collections. Define set $\boldsymbol{R}$:= {1 = Oct$_{(2,3)}$, 2 = Oct$_{(1,2)}$, 3 = Oct$_{(0,1)}$} and group $J := (\boldsymbol{R}, C_3)$. All µBks share an invariant permutation, r , mapping Oct$_{(2,3)}$ ↦ Oct$_{(1,2)}$ ↦ Oct$_{(0,1)}$, shown in Figure 10.



i := (1)(2)(3)
r^{-1} : = (123)

Figure 10. μBk group *J*: *r*

Therefore, the group that represents the actions of *P* and *J* in each μ_nBk is the semidirect product

$W := P \rtimes J$.

Definition 18. Semidirect product

> The notion of a semidirect product of two groups generalizes the idea of a direct product. Let *H* and *K* be groups and suppose that we have an action of *H* on *K* which respects the group structure on *K*; so for each $x \in H$ the mapping $u \longmapsto u^x$ is an automorphism of *K*. Put
> $$G := \{(u,x) \mid u \in K, x \in H\}$$
> and define a product on *G* by
> $$(u,x)(v,y) := (uv^{v^{-1}}, xy)$$
> for all $(u,x)(v,y) \in G$.
> It is readily seen that *G* contains subgroups $H^* := \{(i,x) \mid x \in H\}$ and $K^* := (u,i) \mid u \in K\}$ which are isomorphic to *H* and *K*, respectively, and that $G = K^*H^*$ and $K^* \cap H^* = i$. Moreover, K^* is normal in *G* and the way H^* acts on K^* by conjugation reflects the original action of *H* on *K*… We call *G* the semidirect product of *K* by *H* and shall use the notation $K \rtimes H$ to denote *G*. [In the notation ⋊, the open side of the symbol is directed toward the normal].[127]

The group *J* models three elements. As such, the orbits of *W* are of order ≤ 3. Meaning, the action of the semidirect product stabilizes one element in ***N***. Symmetries of the square no longer apply as there is no rigid motion that stabilizes a single point. However, the full symmetry group of the tetrahedron, isomorphic to the alternating group on four elements, written A_4, can produce permutations on four elements with a single stabilized point.

[127] Dixon and Mortimer, 44-5.

Definition 19. Alternating group

Every permutation in S_n, $n > 1$, is a product of 2-cycles.

$$i := (12)(12)$$
$$(12345) = (15)(14)(13)(12)$$
$$(1632)(457) = (12)(13)(16)(47)(45)$$

> The permutation (12345) is expressed as an even number of 2-cycles; (1632)(457) as an odd number of 2-cycles. The group of even permutations of n elements is the alternating group of degree n, [written A_4].[128]

A_4 has the presentation,

$$\langle a, b, c \mid a^4 = b^3 = c^2 = abc = i \rangle.$$

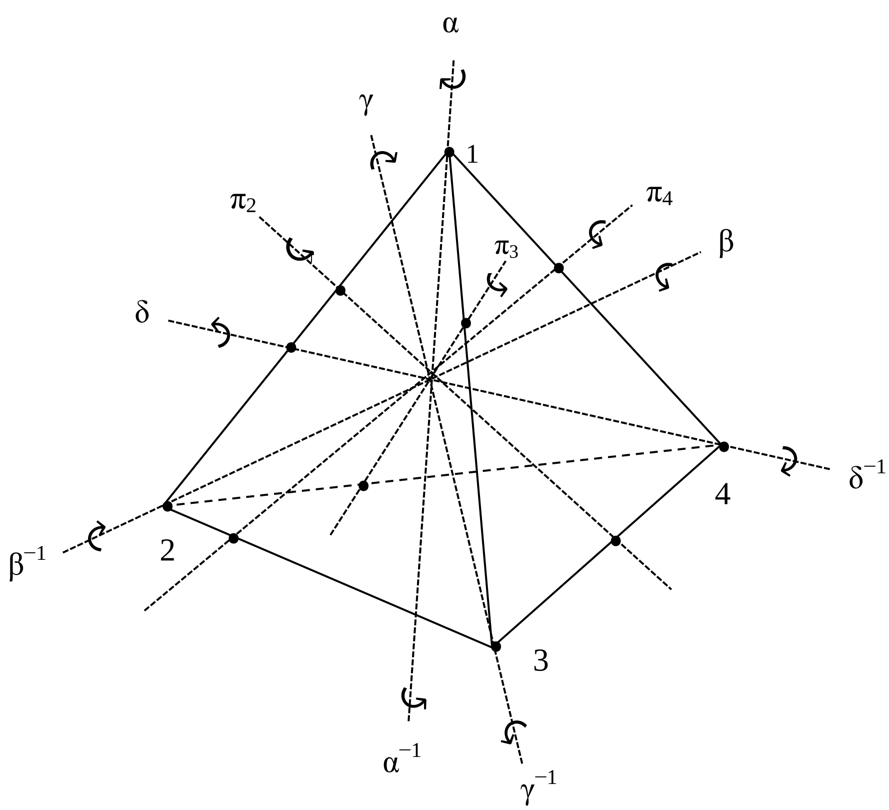


Figure 11. Tetrahedral symmetry, A_4

[128] Gallian, 101-5.

Define the set $\boldsymbol{X}$ as the permutations of P, $\boldsymbol{X} := \{1 = i, 2 = r, 3 = r^2, 4 = r^{-1}\}$, to show members of $W \in A_4$ in cyclic notation, in Example 28. Greek script denotes the A_4 permutations. μBks one and two hold dedicated A_4 permutations, i and π_2 respectively, followed by μBks thee and four that share a single permutation, π_4. This takes us half way through the composition. μBks five and six again hold dedicated permutations, γ^{-1} and α^{-1} respectively, followed by μBks seven and eight that share a permutation, γ^{-1}. The A_4 Cayley table, shown as Table 2, confirms that the product of π_2 by a permutation contained in μBks five though eight produces the other permutations in μBks five through eight: $\pi_2\, \gamma^{-1} = \alpha^{-1}$; $\pi_2\, \alpha^{-1} = \gamma^{-1}$.

Table 2. A_4 Cayley table

	i	π_2	π_3	π_4	δ	α	γ	β	δ^{-1}	β^{-1}	α^{-1}	γ^{-1}
$i = (1)(2)(3)(4)$	i	π_2	π_3	π_4	δ	α	γ	β	δ^{-1}	β^{-1}	α^{-1}	γ^{-1}
$\pi_2 = (12)(34)$	π_2	i	π_4	π_3	α	δ	β	γ	β^{-1}	δ^{-1}	γ^{-1}	α^{-1}
$\pi_3 = (13)(24)$	π_3	π_4	i	π_2	γ	β	δ	α	α^{-1}	γ^{-1}	δ^{-1}	β^{-1}
$\pi_4 = (14)(23)$	π_4	π_3	π_2	i	β	γ	α	δ	γ^{-1}	α^{-1}	β^{-1}	δ^{-1}
$\delta = (123)$	δ	β	α	γ	δ^{-1}	γ^{-1}	β^{-1}	α^{-1}	i	π_4	π_2	π_3
$\alpha = (243)$	α	γ	δ	β	β^{-1}	α^{-1}	δ^{-1}	γ^{-1}	π_2	π_3	i	π_4
$\gamma = (142)$	γ	α	β	δ	α^{-1}	β^{-1}	γ^{-1}	δ^{-1}	π_3	π_2	π_4	i
$\beta = (134)$	β	δ	γ	α	γ^{-1}	δ^{-1}	α^{-1}	β^{-1}	π_4	i	π_3	π_2
$\delta^{-1} = (132)$	δ^{-1}	α^{-1}	γ^{-1}	β^{-1}	i	π_3	π_4	π_2	δ	γ	β	α
$\beta^{-1} = (143)$	β^{-1}	γ^{-1}	α^{-1}	δ^{-1}	π_2	π_4	π_3	i	α	β	γ	δ
$\alpha^{-1} = (234)$	α^{-1}	δ^{-1}	β^{-1}	γ^{-1}	π_3	i	π_2	π_4	γ	δ	α	β
$\gamma^{-1} = (124)$	γ^{-1}	β^{-1}	δ^{-1}	α^{-1}	π_4	π_2	i	π_3	β	α	δ	γ

Although not triadic in nature, the melody acts in conjunction with the triads and offers the opportunity to develop further the discussion on A_4. “Hotel Vamp’s” melody features a single pitch for each μSeg. The melody’s structure is a chromatic descending fourth progression spanning D♭ to A♭, which takes two μBks to complete. Within each μSeg,the melodic pitch holds a different relationship with each triad in $\boldsymbol{N}^{[x]}$. Accepting enharmonic equivalence, these relationships define $\boldsymbol{F} := \{4, 9, \Delta 7, \flat 6\}$, the set in $L := (\boldsymbol{F}, C_4)$.

Example 30. $L := (\boldsymbol{F}, C_4)$

Melodic Relation	as $\in \mathbb{Z}_4$	Permutation	Cyclic Notation	$x \in L$
(4,9,Δ7,♭6)	(1,2,3,4)	4 9 Δ7 ♭6 4 9 Δ7 ♭6	( )	i
(♭6,4,9,Δ7)	(4,1,2,3)	4 9 Δ7 ♭6 ♭6 4 9 Δ7	(1234)	r
(Δ7,♭6,4,9)	(3,4,1,2)	4 9 Δ7 ♭6 7 ♭6 4 9	(13)(24)	r^2
(9,Δ7,♭6,4)	(2,3,4,1)	4 9 Δ7 ♭6 9 7 ♭6 4	(1432)	r^{-1}

Define the semidirect product $Z := L \rtimes J$ and the set $\boldsymbol{V} := \{1 = i, 2 = r, 3 = r^2, 4 = r^{-1}\}$. Permutations of L appear as integers to facilitate viewing $Z \in A4$ and to model the order 3 orbits generated by Z. Members of Z derive from products of members in W, situated below Z in Example 28.

Single Z members occupy each µBk, up to µBks seven and eight, where the A_4 exchange π_3 occupies two µBks. π_3 is of importance. It answers the set of exchanges π_2 and π_4 of W in µBks two through four and acts as a permutational model for the melodic group actions. Define hyper-µBks as the union of two µBks. The products of Z permutations held in each hyper-µBk, prior to µBk seven, confirms the importance of π_3, as the products of the A_4 permutations in the first three hyper-µBks all equal π_3. The hyper-µBk delineated products $(xy(z)) \in W$ also confirms the importance of π_3.

Single Z members occupy each µBk, up to µBks seven and eight, where the A_4 exchange π_3 occupies two µBks. π_3 is of importance. It answers the set of exchanges π_2 and π_4 of W in µBks two through four and acts as a permutational for the melodic group actions. Define hyper-µBks as the union of two µBks. The products of Z permutations held in each hyper-µBk, prior to μBk seven, confirms the importance of π_3, as the products of the A_4 permutations in the first three hyper-μBks all equal π_3. The hyper-μBk delineated products $(xy(z)) \in W$ also asserts the importance of $\pi_{3.}$

CHAPTER 3. GROUP ACTIONS

3.1. Introduction

In this section, we investigate group actions on the set of consonant triads that derive from the scale collections covered the preceding discussions; we then posit a theory to apply the group actions to musically relevant contexts. Sets are organized by cardinality, (4…8), and are listed in the Scale Roster.

3.2. Scale Roster

The Scale Roster lists the constituent consonant triadic sets contained within nine unique scale classes, which occupy thee scale genres based on how the scale is generated.[129] Two criteria determine the inclusion of a scale within the roster. First, the scale is significant in the accepted jazz compositional or improvisational canon; second, the scale is capable of producing a sufficient number of consonant triads required to investigate symmetries based on group actions. There are however, scales common to jazz that are not included the roster. The bebop and blues scales are examples of scales commonly used in jazz that contain a chromatic trichord (012) subset. These scales exist within the jazz vocabulary for reasons other than harmonic generation. The blues scale $\{\hat{1}, \flat\hat{3}, \hat{4}, \flat\hat{5}, \natural\hat{5}, \flat\hat{7}\}$ categorizes chromatic inflections as "blue notes," in an attempt to approximate a melismatic vocal style. Bebop scales (there are a number of bebop scales, but to illustrate, consider the dominant-seventh bebop scale

[129] For other examples of scale roster type organizational schemes, see, Jeff Pressing, "Towards an Understanding of Scales in Jazz," *Jazz Research* 9 (1979); Phillip Wade Russom, "A Theory of Pitch Organization for the Early Works of Maurice Ravel" (PhD diss., Yale University, 1985); and Dmitri Tymoczko, "Scale Networks and Debussy," *Journal of Music Theory* 48, no.2 (2004).

$\{\hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}, \hat{6}, \flat\hat{7}, \natural\hat{7}\}$) used to reconcile chord tones with rhythmic downbeats.[130] Therefore, although the blues scale contains scale degree ♭5 and the dominant-quality be-bop scale contains scale degree ♮7, harmonies are not generally built on these scale degrees.

Constituent consonant triads therefore include triads that hold to typical harmonic spellings and chords derived from them through enharmonic equivalence; square brackets represent the latter.

Table 3. Scale roster

Genre	Scale: C = $\hat{1}$	Symbol	Constituent Consonant Triads	$\lvert\boldsymbol{X}\rvert$
Generated by i.c.4	Diatonic {C,D,E,F,G,A,B}	$\boldsymbol{D}_{(n)}$	{C$^\Delta$,D$^-$,E$^-$,F$^\Delta$,G$^\Delta$,A$^-$}	6
Synthetic	Real Melodic Minor {C,D,E♭,F,G,A,B}	$\boldsymbol{M}_{(x)}$	{C$^-$,D$^-$,F$^\Delta$,G$^\Delta$}	4
	Real Melodic Minor ♯5 {C,D,E♭,F,G♯,A,B}	$\boldsymbol{M}\text{+}_{(x)}$	{D$^-$,F$^\Delta$,[F$^-$],G♯$^\Delta$,[G♯$^-$]}	5
	Harmonic Minor {C,D,E♭,F,G,A♭,B}	$\boldsymbol{H}_{(x)}$	{C$^-$,F$^-$,G$^\Delta$,A♭$^\Delta$}	4
	Harmonic Major {C,D,E,F,G,A♭,B}	$\boldsymbol{HM}_{(x)}$	{C$^\Delta$,E$^-$,[E$^\Delta$],F$^-$,G$^\Delta$}	5
	Double Harmonic {C,D♭,E,F,G,A♭,B}	$\boldsymbol{DH}_{(x)}$	{C$^\Delta$,D♭$^\Delta$,[D♭$^-$],[E$^\Delta$],E$^-$,[F$^-$]}	6
	Double Harmonic ♯5 {C,D♭,E,F,G♯,A,B}	$\boldsymbol{DH}\text{+}_{(x)}$	{[D♭$^\Delta$],[D♭$^-$],E$^\Delta$,F$^\Delta$,[F$^-$],[A$^\Delta$],A$^-$}	7
Symmetric	Hexatonic {C,C♯,E,F,G♯,A}	$\boldsymbol{Hex}_{(x,y)}$	{D♭$^\Delta$,D♭$^-$,F$^\Delta$,F$^-$,A$^\Delta$,A$^-$}	6
	Octatonic {C,D♭,E♭,E,F♯,G,A,B♭}	$\boldsymbol{Oct}_{(x,y)}$	{C$^\Delta$,C$^-$,E♭$^\Delta$,E♭$^-$,G♭$^\Delta$,G♭$^-$,A$^\Delta$,A$^-$}	8

Constituent consonant triadic set cardinality determines the proper group and geometric structure for modeling the scale. The following subsections address sets based on set cardinality. Following the group notation used above, sets from the Scale Roster take the notation

$$\boldsymbol{X}_n$$

[130] Levine, 170-3.

where $\boldsymbol{X}$ denotes the set, and n the pitch level identification. For diatonic collections, pitch level is given by key signature, (n = number of flats or sharps); for synthetic collections, the first scale degree is provided by x (where the variable x takes the place of variable n) ∊ p.c. (0…11); and for symmetric collections, the first two p.c.s are shown as (x,y).

3.3. Symmetries on 4 Elements

Scales modeled on four elements include the real melodic minor ($\boldsymbol{M}$) and harmonic minor ($\boldsymbol{H}$). Symmetries of the square (2-cube), previously discussed in detail, invoke C_4 and D_8. However, there is another group that we may employ to model a set of 4 elements, the Klein 4-group, also known as the *Viergruppe* (four-group), written V_4.

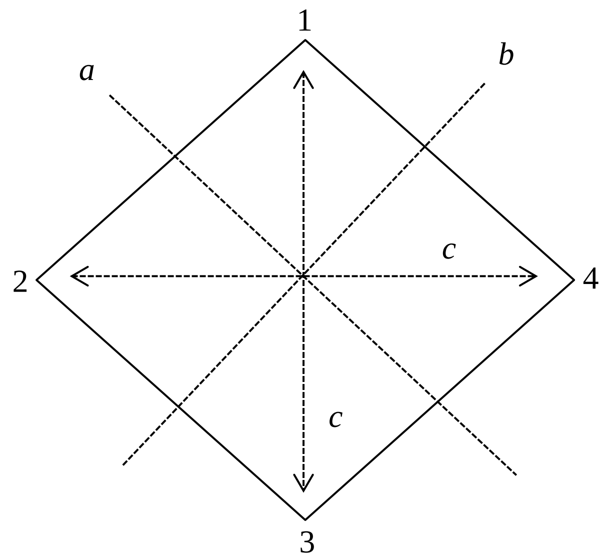
Definition 20. Klein 4-group

"Let $\boldsymbol{J} := \{1,2,3,4\}$ and put $V_4 := \{(i), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. Each group element is its own inverse, and the product of any two of the three nonidenity elements is the third one."[131]

$$\langle A, B, C \mid A^2, B^2, C^2, AB = C = BA \rangle,$$

V_4 is the symmetry of the rhombus, displayed in Figure 12. $V_4 \cong C_2 \times C_2$, $V_4 \lhd A_4$.

[131] W.R. Scott, *Group Theory* (Englewood Cliffs: Prentice-Hall, 1964), 13.



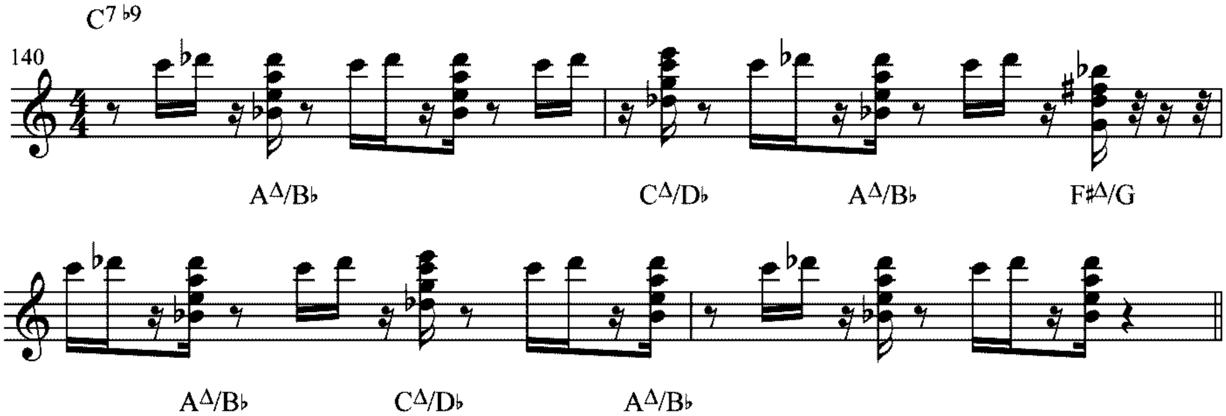
V_4
$i := (1)(2)(3)(4)$
$a := (12)(34)$
$b := (14)(23)$
$c := (13)(24)$

Figure 12. Rhomboidal full symmetry group $\cong V_4$

Musically, the application of V_4 has been explored for decades. Milton Babbitt describes the group of permutations that represent the classic twelve-tone operations, P, R, I, RI as isomorphic to V_4.[132] For our topic, V_4 applies to the organization of triad pairs (drawing upon Campbell's approach). The following example applies V_4 to Pat Martino's triadic gesture from his solo on "Song Bird."[133]

[132] Milton Babbitt, "Twelve-Tone Invariants as Compositional Determinants," in "Problems of Modern Music. The Princeton Seminar in Advanced Musical Studies," special issue, *The Musical Quarterly* 46, no. 2 (1960): 252-3.
[133] Transcription by Jörg Heuser, *Pat Martino*: "*Joyous Lake*" (Mainz: BbArking Munckin Music, 2005), 27.

Example 31. "Song Bird"



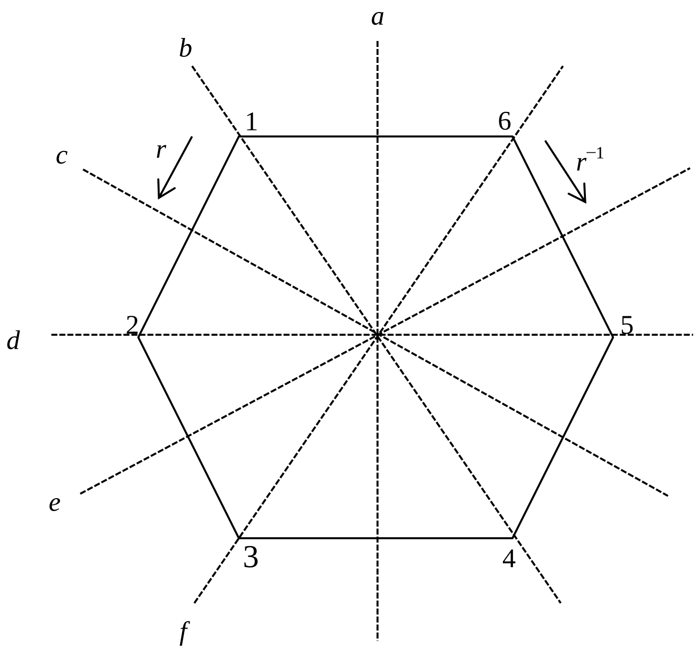
Martino superimposes three major triads {1= A^{Δ}, 2= C^{Δ}, 3=$F\sharp^{\Delta}$} ∊ $\boldsymbol{Oct}_{(0,1)}$, over $C^{7\flat 9}$. The triads present as concatenated exchanges where the third chord of each permutation acts as a pivot into the following exchange.

$$\underbrace{A^{\Delta}\quad C^{\Delta}\quad A^{\Delta}}_{a}\quad \underbrace{F\sharp^{\Delta}\quad A^{\Delta}}_{c}\quad \underbrace{C^{\Delta}\quad A^{\Delta}}_{a}$$

Two exchanges represent the music: (A^{Δ},C^{Δ}) and (A^{Δ},$F\sharp^{\Delta}$). The three triads are well suited for D_6. To model the four major triads in $\boldsymbol{Oct}_{(0,1)}$, define the set $\boldsymbol{N}$:= {1 = A^{Δ}, 2 = C^{Δ}, 3 = $E\flat^{\Delta}$,4 = $F\sharp^{\Delta}$} and consider the permutations suggested by the music as members of A_4, holding $E\flat^{\Delta}$ as the stabilized point. ($\boldsymbol{N}$, V_4) extends this action while maintaining the thematic exchanges. ($\boldsymbol{Oct}_{(x,y)}$, V_4) also applies over the four minor triads in $\boldsymbol{Oct}_{(x,y)}$.

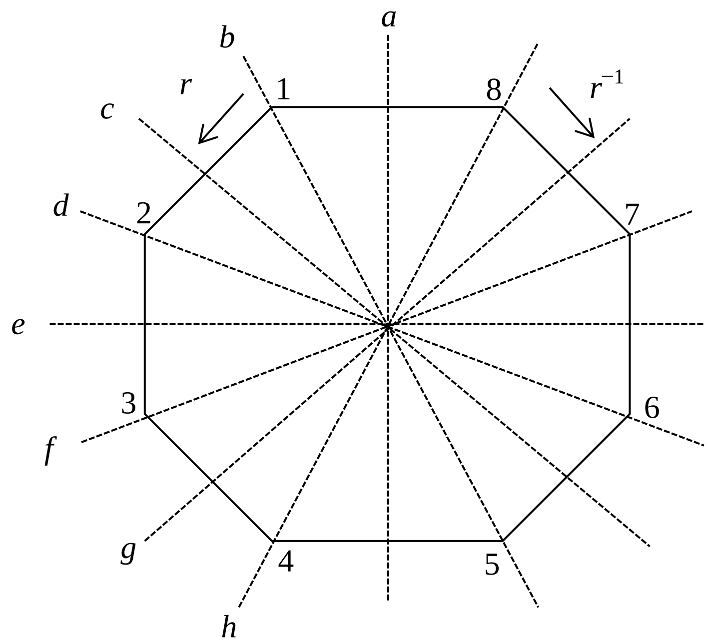
3.4. Symmetries on 6 and 8 Elements

Scales with six triadic elements: diatonic (***D***), double harmonic (***DH***), hexatonic (***Hex***), and the eight-element octatonic (***Oct***) can be modeled as full symmetry groups in two dimensions with D_{12} and D_{16}, respectively.



C_6
i := (1)(2)(3)(4)(5)(6)
r := (123456)
r^2 := (135)(246)
r^3 := (14)(25)(36)
r^4 := (153)(264)
r^{-1} := (165432)
$D_{12} = \langle C_6, f_x \in f_{(a\ldots f)} \rangle$
a := (16)(25)(34)
b := (26)(35)
c := (12)(63)(54)
d := (13)(64)
e := (14)(23)(65)
f := (15)(24)

Figure 13. Hexagonal symmetry ≅ D_{12}

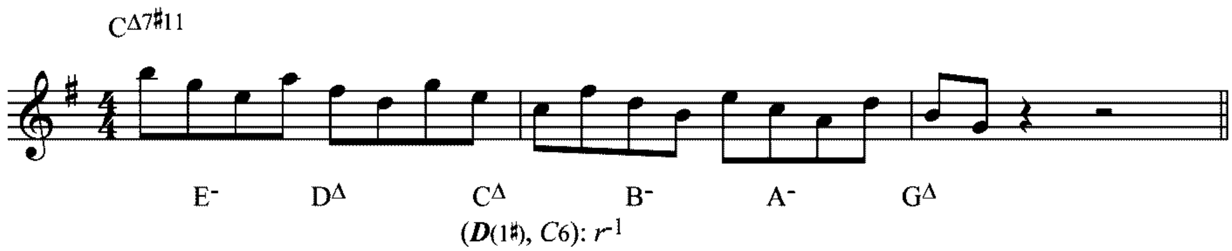


C_8
i := (1)(2)(3)(4)(5)(6)(7)(8)
r := (12345678)
r^2 := (1357)(2468)
r^3 := (14725836)
r^4 := (15)(26)(37)(48)
r^5 := (16385274)
r^6 := (1753)(2864)
r^{-1} := (18765432)
D_{16} = <C$_8$, $f_x \in f_{(a\ldots h)}$>
a := (18)(27)(36)(45)
b := (28)(37)(46)
c := (12)(83)(74)(65)
d := (13)(84)(75)
e := (14)(23)(76)(85)
f := (15)(24)(86)
g := (16)(25)(34)(87)
h := (17)(26)(35)

Figure 14. Octagonal group ≅ D_{16}

Cyclic groups are the only groups included in this study that permute the set in a single orbit (in reference to r and r^{-1}). This action is most noticeable when modeling the diatonic. Musicians can easily discern stepwise motion in this familiar collection. Pedagogically, cyclic groups and dihedral groups are good introductory topics. Garrison Fewell, in his textbook *Jazz Improvisation* describes "diatonic substitution," where triads from the parent diatonic collection are superimposed over a stated harmony, see Example 32.[134] We recognize this as $(\boldsymbol{D}_{(1\sharp)}, C_6)$: r^{-1}.

Example 32. Garrison Fewell's diatonic substitution

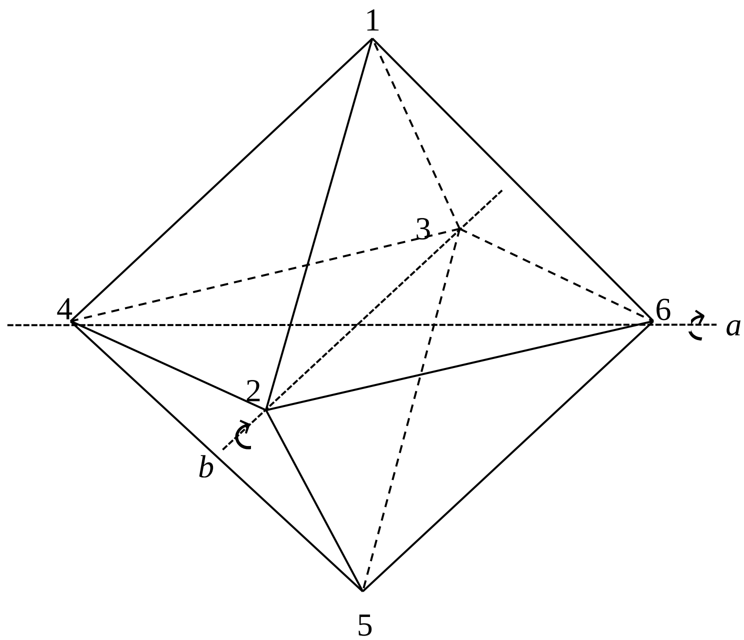


Sets of order 6 and order 8 hold symmetries in three dimensions as well. The octahedral rotational group, denoted O, is the rotations of the octahedron and of the cube, and has the following presentation,

$$<s,t \mid s^2, t^3, (st)^4>.$$

$O \cong S_4$, $O \cong A_4 \rtimes C_2$, $|O| = 24$. O has two generators, $<s,t>$ in the above presentation. The generators of O acting on the two symmetric scales in the scale roster are: $(\boldsymbol{Hex}_{(x,y)}, O)$: $<a,b>$; $(\boldsymbol{Oct}_{(x,y)}, O)$: $<g,h>$.[135] We first consider rotational symmetry of the octahedron acting on $\boldsymbol{Hex}_{(x,y)}$.

[134] Garrison Fewell, *Jazz Improvisation* (n.p.: Garrison Fewell, 1986), 44-7.
[135] Appendix D contains a list of $(\boldsymbol{Hex}_{(3,4)}, O)$ and $(\boldsymbol{Oct}_{(0,1)}, O)$ permutations.



$a := (1352)$

$b := (1654)$

Figure 15. $O := \langle a,b \rangle$

With ($\mathbf{Hex}_{(x,y)}$, O), we gain certain neo-Riemannian transformations ($P = a^{-1}$ and $a^{-1}b^2$), ($L = ab^{-1}a$); and the hexatonic pole[136] (($PLP = LPL$) $= a^2b^3$ and b). We consider the neo-Riemannian transformations and their generative group actions in the analysis of an excerpt from Pat Martino's solo on "Joyous Lake."

Analysis 8. "Joyous Lake," excerpt[137]

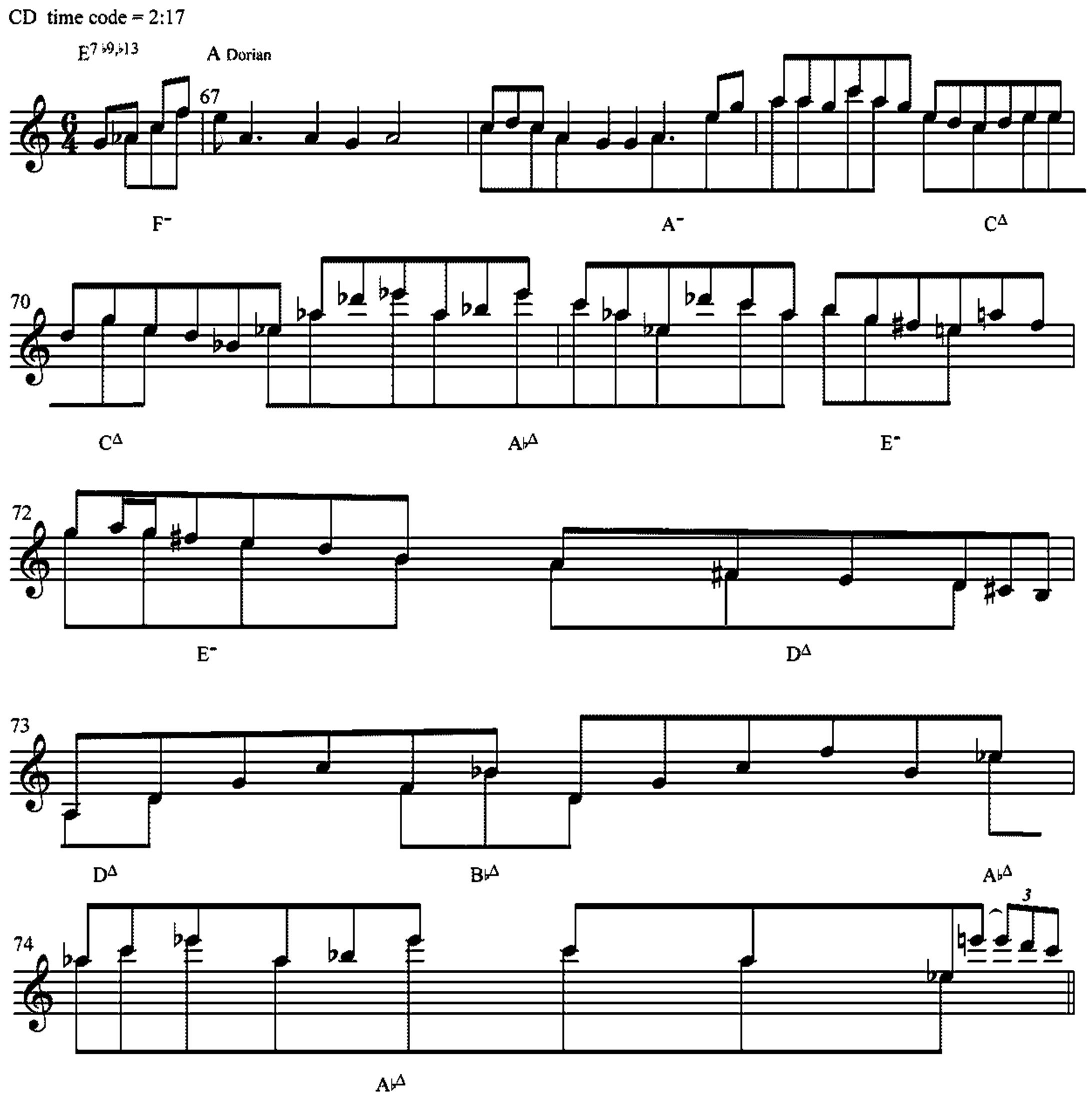
Pat Martino's solo excerpt, taken from "Joyous Lake," is over A Dorian harmony. Martino uses eight concatenated triads from the sets, two copies of $\boldsymbol{Hex}_{(x,y)}$, and $\boldsymbol{D}$, being acted on by three permutations in O. The analysis below addresses these triadic permutations as neo-Riemannian transformations and as actions of O.

---

[136] See Cohn, *Audacious Euphony*, 31.

[137] Transcription by Jörg Heuser, *Pat Martino*: "*Joyous Lake*" (Mainz: BbArking Munckin Music, 2005), 30-1.

Example 33.1. “Joyous Lake,” excerpt



Example 33.2. “Joyous Lake,” triadic analysis

RL R PL PL

F^- A^- C^Δ $A\flat^\Delta$ E^- D^Δ $B\flat^\Delta$ $A\flat^\Delta$

$(\boldsymbol{Hex}_{(0,1)}, O)$: a^2b $(\boldsymbol{D}_{(\emptyset)}, O)$: ab^2 $(\boldsymbol{Hex}_{(3,4)}), O)$: a^2b $(\boldsymbol{Hex}_{(1,2)}, O)$: a^2b

Hexatonic Pole $(\boldsymbol{Hex}_{(3,4)}, O)$: a^2b^3

Martino begins with an F^- triad over V^7/A^-, (E^7). He then moves to an A^- tonic triad, followed by its relative major, C^Δ, that gives way to a ***Hex*** related (*PL*) $A\flat^\Delta$. A hexatonic pole continues hexatonic sonority, transforming A♭$^{\Delta}$ ⟼ E^{-} (this hexatonic pole also includes the final A♭$^{\Delta}$ chord). An interpolated *PL* transforms D$^{\Delta}$ ⟼ B♭$^{\Delta}$, resides within the hexatonic pole's presentation.

Three members of *O* are in play: a^2b, ab^2, and a^2b^3. All three permutations belong to a single equivalency class called a conjugacy class.[138]

Definition 21. Conjugacy class

> Let s and t be members of a group G. s and t are conjugate in G (and call t a conjugate of s) if $xsx^{-1} = t$ for some x in G. The conjugacy class of s is the set cl(s) = $\{xsx^{-1} \mid x \in G\}$. Therefore, the conjugacy class of s is the equivalence class of s under conjugacy. In other words, conjugacy classes are a means by which to partition the group members. Each group member belongs to a single conjugacy class.[139]

O contains five conjugacy classes. As to a^2b, ab^2, a^2b^3, on ($\boldsymbol{Hex}_{(x,y)}$, *O*) and ($\boldsymbol{D}_{(\emptyset)}$, *O*), all are members of cl(*aba*). A Permutational Triadic reading tells us that while Martino utilizes two differing scale genres at four differing pitch levels, being acted upon by three permutations in *O*, we can describe the musical event as a single entity, cl(*aba*) ∈ *O*.

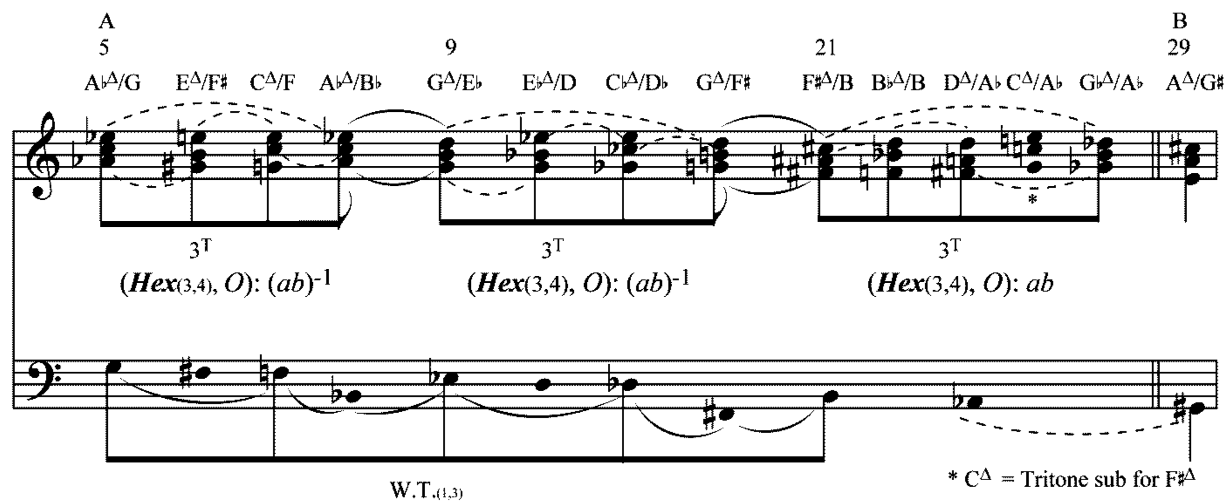
The next analysis of Kenny Wheeler's "Ma Belle Hélène," looks at *O* acting on a single scale genus, the hexatonic. To facilitate the analysis, we return to the technique employed in the Goodrick analysis (Analysis 3) of separating triad-over-bass-note structures into segregated parts, the bass line and the upper-structure triad. "Ma Belle Hélène" also provides an opportunity to apply group actions to a voice-leading scheme, alluding to additional applications of the Permutational Triadic Approach.

138 For a listing of conjugacy classes of *O*, see Appendix D.
139 Gallian, 395-6.

Analysis 9. “Ma Belle Hélène”

“Ma Belle Hélène” is a 50-measure composition in AB form with an introduction. Except for three chords, the $A\flat^{sus}$ in mm. 22-24, the A^{sus} in m.81, and the D^{-} that acts as a final point of arrival, all harmonies take the form of triads-over-bass-notes (as in the Goodrick example).[140] Formal sections A and B share nearly identical harmonies, albeit section B is at T_1 to section A. As such, the present analysis focuses on the A section.[141]

Example 34. “Ma Belle Hélène,” graphic analysis



The first step in the analysis is to determine if the bass notes are acting as a set-union member with the upper structure triad, or if the bass line is an independent musical structure. The latter prevails. In the span of mm. 5-21, the bass line projects a descending whole-tone 6th progression. One note separates each whole-tone member. The extra note’s function alternates between being a descending chromatic passing tone and a pitch related by cycle-5 motion to both the note before it and the note after, the latter being reminiscent of an extended dominant pattern.

---

[140] On the recording, these sus chords are played as (027) trichords ((A♭, D♭, E♭) and (A, D, E)). Dominant seventh sus 9 chords can present as upper structure major triads built on roots a whole-step below the bass note, i.e. $G\flat^{\Delta}/A\flat$. For analytical purpose, A♭ sus, the last harmony of the last 3^T set is realized an upper structure to show its relationship to the first chord of the last 3^T set.

[141] Appendix B contains the lead sheet in the composer’s hand.

The introduction of the pitch A♭, an enharmonically reinterpreted anticipation of the first p.c. in the formal B section, breaks the whole-tone projection.

Upper structure triads are delineated into three 3^{T} sets of order 4, shown with beams in Example 34. In each set, the start and end triads are the same. The end chords in the first two sets act as ^{sub}V dominant-action chords to the start chord of the next set. The group O actions on these sets are as follows: set 1= ($\boldsymbol{Hex}_{(3,4)}$, O): $(ab)^{-1}$; set 2 = ($\boldsymbol{Hex}_{(2,3)}$, O): $(ab)^{-1}$; set 3 = ($\boldsymbol{Hex}_{(1,2)}$, O): ab. The root motion in sets 1 and 2 takes the form of directed i.c. −4, and in set 3, directed i.c. +4. The reason is apparent; the group action $(ab)^{-1}$ on sets 1 and 2 is the inverse of the group action (ab) on set 3, thus reversing the directed i.c. direction in p.c. space.

Voice leading holds one common tone between each adjacent triad within a set. Define the voices as V1 = top voice, V2 = middle voice, and V3 = bottom voice. In sets 1 and 2, the permutation of the voices holding the common tone is (312) = C_3: r. In set 3 it is (213) = C_3: r^{-1}. This process echoes the inverse relation held by the action of O on the triadic sets.

Chord/ scale determination could take the form of set inclusion, where each upper-structure triad and the associated bass-note combination forms a tetrachord and the scales that contain each particular tetrachord as a subset are available chord/scale choices. With this approach, we must redefine the chord/scale choice for each change in harmony. The Triadic Permutational Approach, on the other hand, allows each set of four triads to take the chord/scale that is based on the scale used to define the group. In the present case, that is a hexatonic collection for each triadic set. This approach frees the soloist from the abruptly shifting scale genres and transpositional levels associated with realizing a separate chord/scale for each triad-over-bass-note.

There is a mathematical relationship between the hexatonic and octatonic collections beyond that of symmetry in chromatic space. The octahedron's six vertices identify with the six sides of a 3-cube: they form a geometric dual, shown in Figure 16. Octatonic set elements = ω*n*; hexatonic set elements = η*n*. Therefore, the group used to model the hexatonic collection is isomorphic to the group used to model the octatonic collection, although the size of the sets used to define the group are of different order, $(\boldsymbol{Hex}_{(x,y)}, O) \cong (\boldsymbol{Oct}_{(x,y)}, O)$.

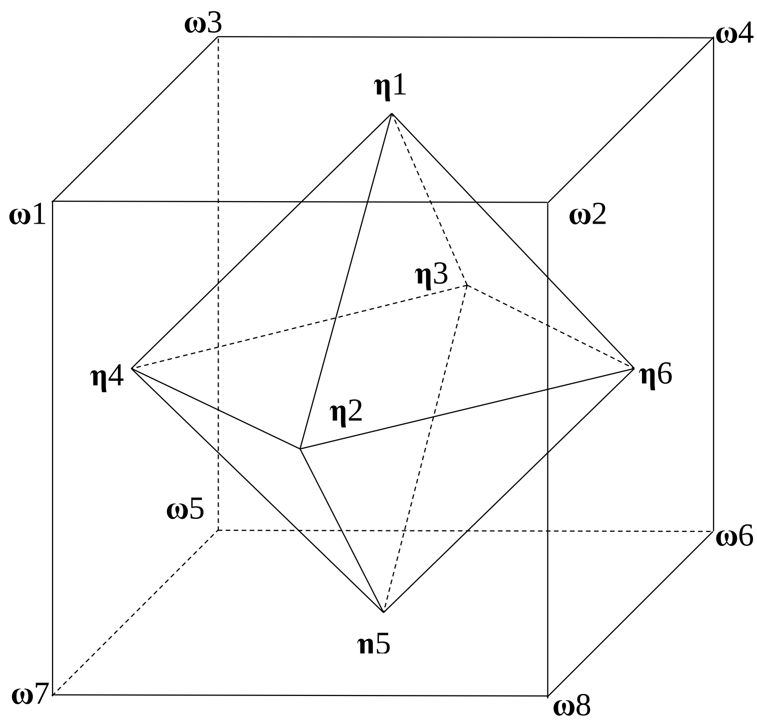
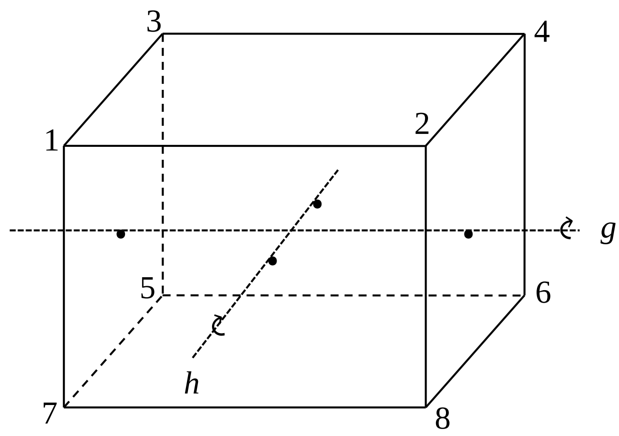


Figure 16. Octahedral dualism

Two group actions generate the rotational symmetry of the cube, $(\boldsymbol{Oct}_{(x,y)}, O)$: $\langle g,h \rangle$. Note that generator *g* equals the generator *a* in $(\boldsymbol{Hex}_{(x,y)}, O)$ and generator *h* equals generator *b* in $(\boldsymbol{Hex}_{(x,y)}, O)$.



i := (1)(2)(3)(4)(5)(6)(7)(8)
g := (1357)(2468)
h := (1287)(3465)

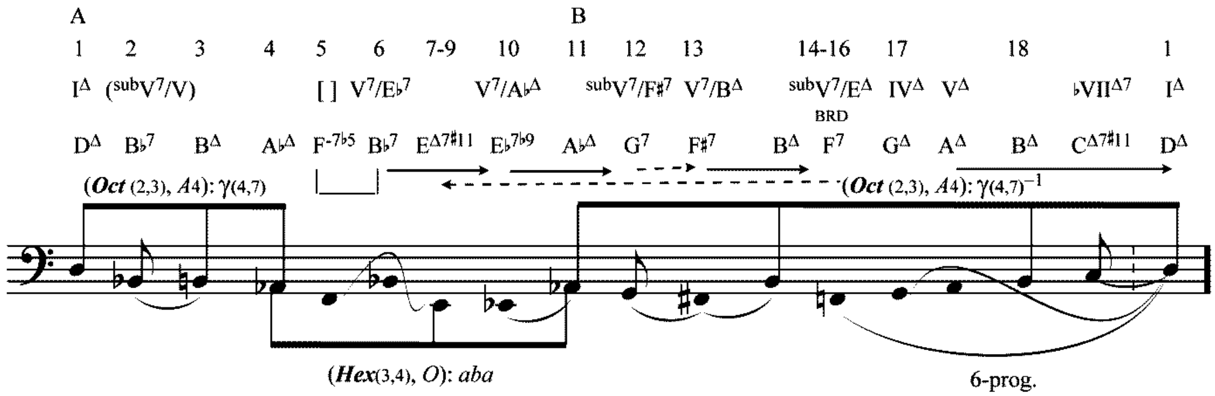
Figure 17. ($\boldsymbol{Oct}_{(x,y)}$, O): <g,h>

We now turn to Joe Henderson's composition "Punjab" to use the geometric duality in a musical analysis.

Analysis 10. "Punjab"

Joe Henderson's "Punjab" is an example of a composition where multiple tonic systems form the harmonic middle ground. The eighteen-bar form contains two unique tonic systems from the sets $\boldsymbol{Oct}_{(2,3)}$ and $\boldsymbol{Hex}_{(3,4)}$.The sets occupy the following measures, $\boldsymbol{Oct}_{(2,3)}$ = mm. 1-4 and 11-1; $\boldsymbol{Hex}_{(3,4)}$ = in mm. 4-11. Example 35 shows the interaction of the two symmetric sets graphically.

Example 35. “Punjab”



Oct$_{(2,3)}$ opens and closes the composition. Because the form is cyclic, the end chords must return the music to the top of the form.[142] This is achieved by the 4^T system. Measures 1-4 contain (D^Δ, B^Δ, $A\flat^\Delta$); mm. 11-1 hold ($A\flat^\Delta$, B^Δ, D^Δ), in which the final D^Δ represents the beginning of a new chorus. Let us consider the set of four major triads in ***Oct***$_{(x,y)}$. Since the orbits of triads in “Punjab” are of order 3, there exists a stabilized point, in this case, F^Δ. This group action belongs to A_4, the alternating group on four elements. To show this mapping, a permutation isomorphic reordering of the triads on the cube’s vertices defines the vertices according to chord quality.[143]

Definition 22: Permutation isomorphic

> Two permutation groups, say $G \leq Sym(\Omega)$ $[S_n]$ and $H \leq Sym(\Omega')$ $[S_n']$ are called permutation isomorphic if there exists a bijection $\lambda : \Omega \to \Omega'$ and a group isomorphism $\psi : G \to H$ such that
>
> $$\lambda(\alpha^x) = \lambda(\alpha)^{\psi(x)} \text{ for all } \alpha \in \Omega \text{ and } x \in G.$$
>
> Essentially, this means that the groups are “the same” except for the labeling of the points.[144]

[142] This refers to the cyclic nature of jazz compositions where the form repeats for solos and the recapitulation of the melody.

[143] Symmetric scales do not technically hold to the theory of scale degrees; however, we require the identification of pitch level in order to define permutations. For symmetric scale, we adopt carat notation of “scale degrees” orientating $\hat{1}$ as the pitch lying above, and closest to C = 0.

[144] Dixon and Mortimer, 17.

$S^{\Delta} := \{1 = \hat{1}^{\Delta}, 4 = \flat\hat{3}^{\Delta}, 5 = \mathrm{TT}^{\Delta}, 8 = \hat{6}^{\Delta}\}$; $S^{-} := \{2 = \hat{1}^{-}, 3 = \flat\hat{3}^{-}, 6 = \mathrm{TT}^{-}, 7 = \hat{6}^{-}\}$. The ***Oct***$_{(1,2)}$ ordering is as follows: ***Oct***$_{(2,3)}$:= ($1 = D^{\Delta}$, $4 = F^{\Delta}$, $5 = A\flat^{\Delta}$, $8 = B^{\Delta}$)($2 = D^{-}$, $3 = F^{-}$, $6 = A\flat^{-}$, $7 = B^{-}$). Two tetrahedra inscribed within a cube depict $A_4 \lhd$ (**Oct**$_{(x,y)}$, O) geometrically.

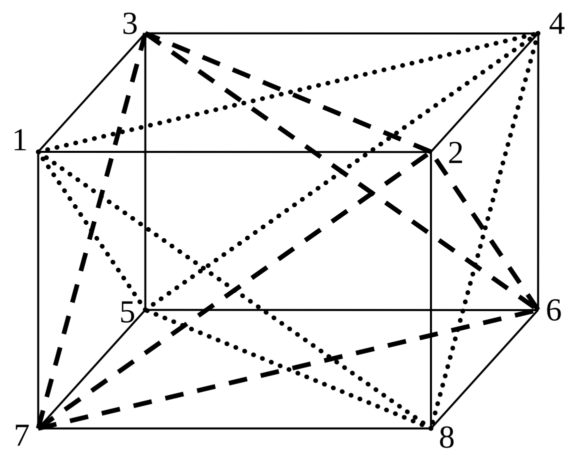


Figure 18. Geometric modeling of $A_4 \lhd$ (***Oct***$_{(1,2)}$, O)

Table 4. Alternating group $A_4 \lhd$ (***Oct***$_{(1,2)}$, O).

	$x \in O$		Triadic Permutation		Triadic Permutation
Group Member	i	i	()	i	()
	$g^{-1}h$	$\gamma_{(1,6)} = (1)(485)$	$(D^{\Delta})(F^{\Delta},B^{\Delta},A\flat^{\Delta})$	$\delta_{(1,6)} = (6)(273)$	$(A\flat^{-})(D^{-},B^{-},F^{-})$
	$(g^{-1}h)^{-1}$	$\gamma_{(1,6)}^{-1} = (1)(458)$	$(D^{\Delta})(F^{\Delta},A\flat^{\Delta},B^{\Delta})$	$\delta_{(1,6)}^{-1} = (6)(237)$	$(A\flat^{-})(D^{-},F^{-},B^{-})$
	$(gh^{-1})^{-1}$	$\gamma_{(4,7)} = (4)(185)$	$(F^{\Delta})(D^{\Delta},B^{\Delta},A\flat^{\Delta})$	$\delta_{(4,7)} = (7)(263)$	$(B^{-})(D^{-},A\flat^{-},F^{-})$
	gh^{-1}	$\gamma_{(4,7)}^{-1} = (4)(158)$	$(F^{\Delta})(D^{\Delta},A\flat^{\Delta},B^{\Delta})$	$\delta_{(4,7)}^{-1} = (7)(236)$	$(B^{-})(D^{-},F^{-},A\flat^{-})$
	gh	$\gamma_{(8,3)} = (8)(145)$	$(B^{\Delta})(D^{\Delta},F^{\Delta},A\flat^{\Delta})$	$\delta_{(8,3)} = (3)(267)$	$(F^{-})(D^{-},A\flat^{-},B^{-})$
	$(gh)^{-1}$	$\gamma_{(8,3)}^{-1} = (8)(154)$	$(B^{\Delta})(D^{\Delta},A\flat^{\Delta},F^{\Delta})$	$\delta_{(8,3)}^{-1} = (3)(276)$	$(F^{-})(D^{-},B^{-},A\flat^{-})$
	hg	$\gamma_{(5,2)} = (5)(148)$	$(A\flat^{\Delta})(D^{\Delta},F^{\Delta},B^{\Delta})$	$\delta_{(5,2)} = (2)(367)$	$(D^{-})(F^{-},A\flat^{-},B^{-})$
	$(hg)^{-1}$	$\gamma_{(5,2)}^{-1} = (5)(184)$	$(A\flat^{\Delta})(D^{\Delta},B^{\Delta},F^{\Delta})$	$\delta_{(5,2)}^{-1} = (2)(376)$	$(D^{-})(F^{-},B^{-},A\flat^{-})$
	$(h^{g})^{2}$	$\pi_1 = (14)(85)$	$(D^{\Delta},F^{\Delta})(B^{\Delta},A\flat^{\Delta})$	$\rho_1 = (23)(67)$	$(D^{-},F^{-})(A\flat^{-},B^{-})$
	h^{2}	$\pi_2 = (18)(45)$	$(D^{\Delta},B^{\Delta})(F^{\Delta},A\flat^{\Delta})$	$\rho_2 = (27)(36)$	$(D^{-},B^{-})(F^{-},A\flat^{-})$
	g^{2}	$\pi_3 = (15)(48)$	$(D^{\Delta},A\flat^{\Delta})(F^{\Delta},B^{\Delta})$	$\rho_3 = (26)(37)$	$(D^{-},A\flat^{-})(F^{-},B^{-})$

The A_4 permutations in "Punjab" are: mm. 1-4 = $\gamma_{(4,7)}$, and mm. 11-1 = $\gamma_{(4,7)}{}^{-1}$. Note the harmonic palindrome generated by the inversional relationship, (D^{Δ},B^{Δ},$A\flat^{\Delta}$),($A\flat^{\Delta}$,B^{Δ},D^{Δ}), among members of $\boldsymbol{Oct}_{(2,3)}$.

An exchange, ($A\flat^{\Delta}$,E^{Δ}), ($\boldsymbol{Hex}_{(3,4)}$, O): *aba* separates the two $\boldsymbol{Oct}_{(2,3)}$ statements, thus defining "Punjab's" global group action, $aba^{\gamma_{(4,7)}}$, a conjugation agreeing with ($\gamma_{(4,7)}$)(*aba*)($\gamma_{(4,7)}{}^{-1}$). The $A\flat^{\Delta}$ chord plays an interesting role. It belongs to both $\boldsymbol{Oct}_{(2,3)}$ and $\boldsymbol{Hex}_{(3,4)}$ and acts as a pivot chord between the hexatonic and octatonic collections. The remaining harmonies act as either dominant-action chords, which are foreground embellishments of the tonic system middle ground, or act as members of the turnaround IV^{Δ}—V^{Δ}—$\flat VII^{\Delta}$ [model interchange]—I^{Δ} in mm. 17-1 (the use of measure 1, signifies the return to the top of the form).

A discussion of the octahedral full symmetry group, *Oh*, closes this subsection. As with the previous dihedral groups, the full symmetry group of the octahedron and the full symmetry group of the cube can be generated by a set of rotations and a reflection. *Oh* has the representation,

$$\langle abc \mid a^4 = b^3 = c^2 = abc \rangle.$$

Group *Oh* is of order 48, and is isomorphic to $S_4 \times C_2$. The figures below show *Oh* as symmetries of the octahedron and of the cube. The reflection is through the point of origin in both figures. Appendix D.6-7 contains a list of permutations generated by reflection for ($\boldsymbol{Hex}_{(3,4)}$, *Oh*) and ($\boldsymbol{Oct}_{(0,1)}$, *Oh*).

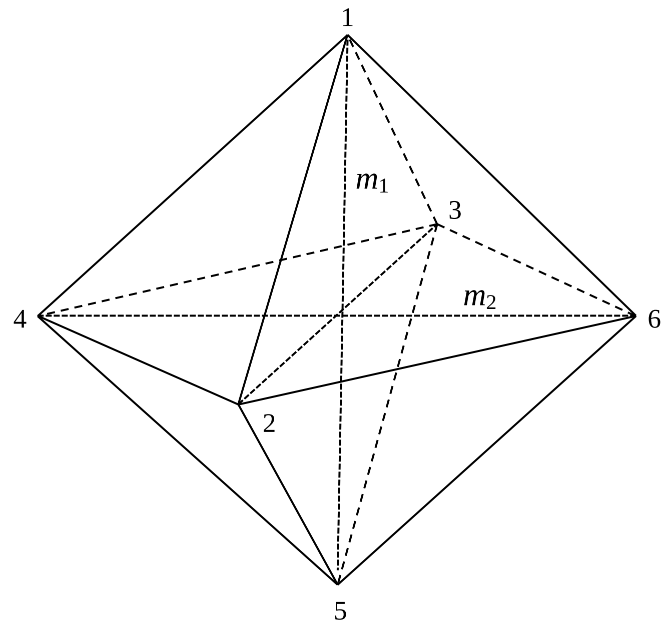


Figure 19.1. Octahedral full symmetry group, tetrahedron

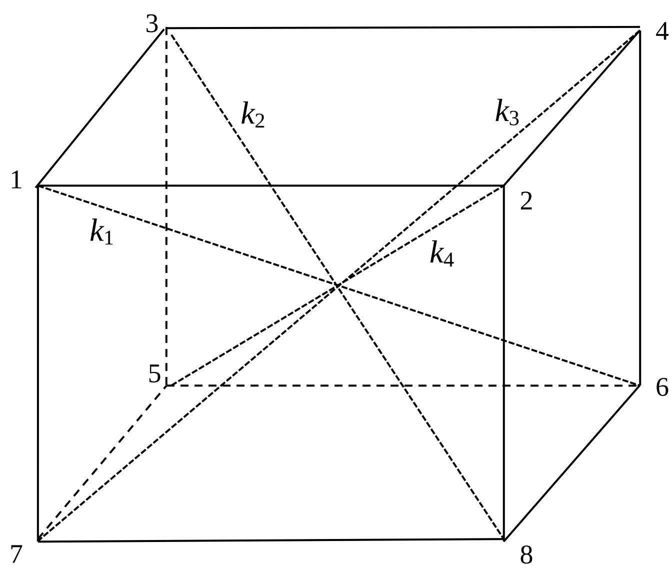
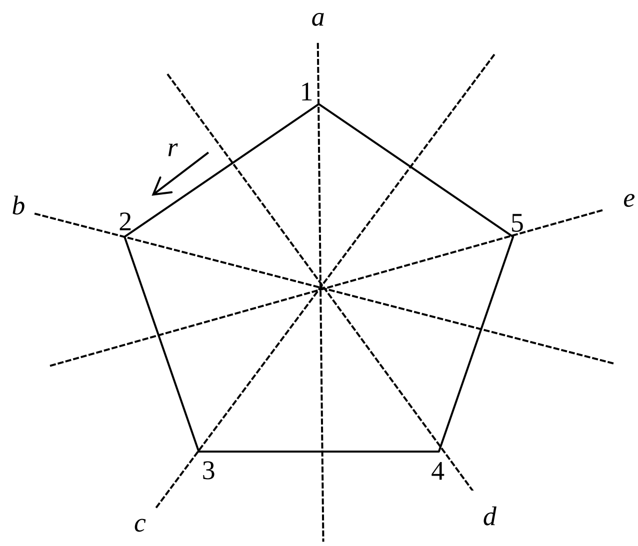


Figure 19.2. Octahedral full symmetry group, cube

3.5. Symmetries on 5 and 7 Elements: *p*-groups

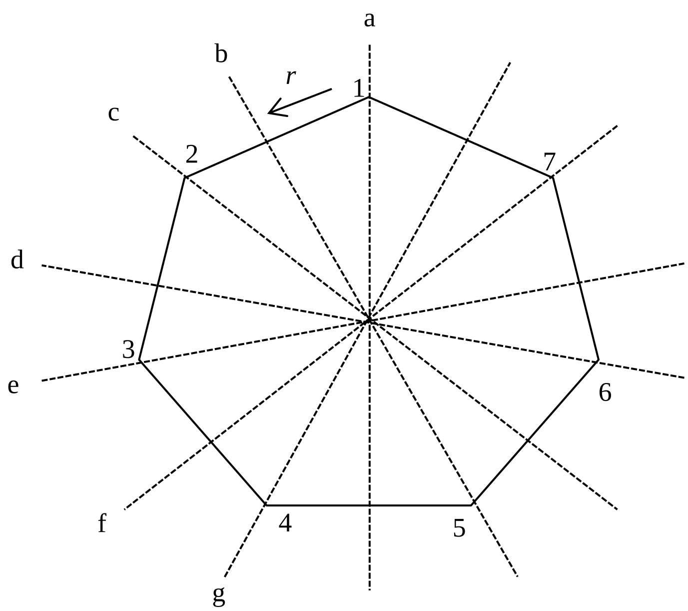
Scales modeled as five elements include real melodic minor ♯5 (***M***+) and harmonic major (***HM***). The double harmonic ♯5 ($\boldsymbol{DH}_{+}$) requires seven elements. Groups of prime order, or powers of primes are called *p*-groups and have as subgroups only the trivial subgroup (*i*) and the whole group itself, when working in Euclidian space.[145] Available groups to illustrate *p*-groups are C_5, C_7, and the addition of a reflection to each of these groups to make their dihedral extensions. The latter relies the semidirect product $C_n \rtimes C_2 \cong D_{2n}$ to generate D_{10} and D_{14}, shown in the following examples.

[145] See, Gallian, 396-7; Moore, 123-7.



C_5
i := (1)(2)(3)(4)(5)
r := (12345)
r^2 := (13524)
r^3 := (14253)
r^{-1} := (15432)
$D_{10} = \langle C_5, f_x \in f_{(a \ldots e)} \rangle$
a := (25)(34)
b := (13)(54)
c := (15)(24)
d := (12)(53)
e := (14)(23)

Figure 20. Pentagonal full symmetry group $\cong D_{10}$



C_7
$i := (1)(2)(3)(4)(5)(6)(7)$
$r := (1234567)$
$r^2 := (1357246)$
$r^3 := (1473625)$
$r^4 := (1526374)$
$r^5 := (1642753)$
$r^6 := (1765432)$
$D_{14} = \langle C_7, f_x \in f_{(a\ldots g)} \rangle$
$a := (27)(36)(45)$
$b := (13)(47)(56)$
$c := (15)(24)(67)$
$d := (17)(26)(35)$
$e := (12)(37)(46)$
$f := (14)(23)(57)$
$g := (16)(25)(34)$

Figure 21. Septagonal full symmetry group

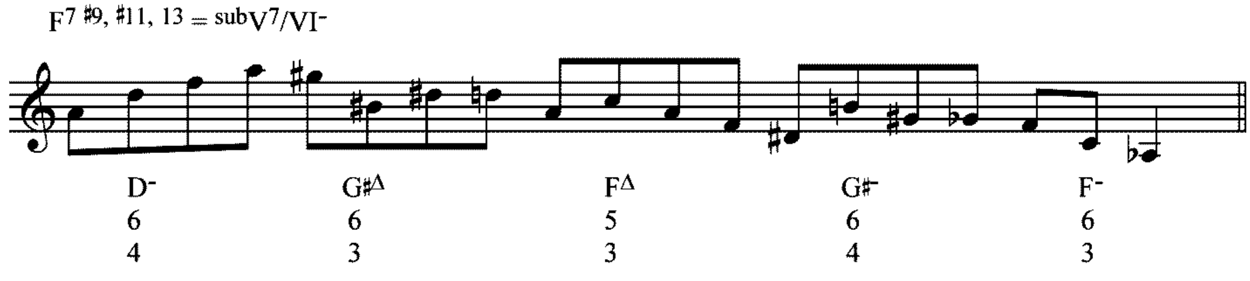
CHAPTER 4. APPLICATION

4.1. *p*-group Application

To illustrate the use of p-groups, we turn to two examples that use sets of cardinality five (real melodic minor #5, ***M***+) and cardinality seven (double harmonic #5, ***DH***+) to increase the amount of altered tensions for a given class of functional harmony. While the focus here is on collections that support p-groups, the steps by which to increase harmonic tension apply to any functional harmony class.

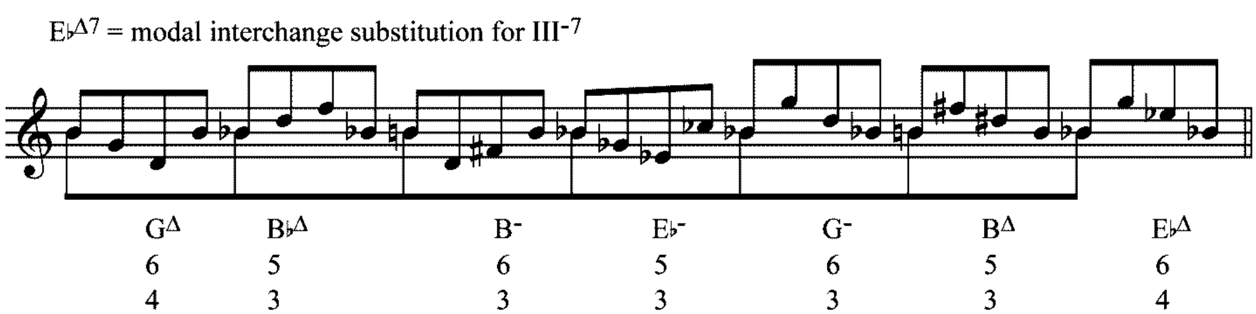
First consider the functional harmony class of $^{sub}V^7/X$. $^{sub}V^7$ chords take Lydian ♭7 as their parent chord/scale, and thus usually carry {9, ♯11, 13} as available tensions. To include ♯9 as an altered tension, we consult appendix A.2.1 and find that the harmony built on the fourth scale degree of ***RM***+ forms a dominant seventh ♯9, ♯11, 13 chord. Let us assume the key of G major where we will work with F^7, $^{sub}V^7/VI^{-7}$, with the chord/scale choice of ***RM***+$_{(0)}$, which generates the set {1 = D–, 2 = F$^{\Delta}$, 3 = F^{-}, 4 = G♯$^{\Delta}$, 5 = G♯$^{-}$}, the constituent consonant triads. Example 36 shows one possibility of ***RM***+ over a $^{sub}V^7$ that incorporates Garzone's approach (forbidding invariant triadic inversional position). The permutation is (***RM***+, C_5): r^3.

Example 36. (***RM***+$_{(0)}$, C_5): r^3 over $^{sub}V^7/VI^{-7}$



For the second example, let us increase the amount of altered tensions through double modal interchange. Retain the assumption of the key of G major and reharmonize $E^{-7} = III^{-7}$ as ♭$III^{\Delta 7}$, a chord borrowed from the parallel Aeolian. The parent chord/scale choice for ♭$III^{\Delta 7}$ is Lydian, providing available tensions 9, ♯11. We could now choose to invoke melodic minor ($\boldsymbol{M}_{(2)}$) to change the quality to major seventh ♯5. If the double harmonic ♯5 scale is the chord/scale choice, we obtain a major seventh 9, ♭13 chord (built on the sixth scale degree, see appendix A.2.5). ♭13 can be enharmonically reinterpreted as ♯5 allowing the musician to toggle between the natural and altered image of the chordal fifth. Define the set of constituent consonant triads from $\boldsymbol{DH}+_{(6)}$ as {1 = G^{Δ}, 2 = G^{-}, 3 = B♭$^{\Delta}$, 4 = B^{Δ}, 5 = B^{-}, 6 = E♭$^{\Delta}$, 7 = E♭$^{-}$}. Example 37 shows a possible superimposition over an altered ♭$III^{\Delta 7}$ acting as a double modal interchange chord for III^{-7}. The permutation is $(\boldsymbol{DH}+_{(6)}, C_7)$: r^2.

Example 37. $(\boldsymbol{DH}+_{(6)}, C_7)$: r^2 acting over ♭$III^{\Delta 7\,\flat 6}$



4.2. 3^T Systems Revisited

To model triadic permutations over a standard but unique chord progression, we consider the 3^T system derived from $\boldsymbol{Hex}_{(2,3)}$, the same tonic system found in “Giant Steps.” The musical goal is to identify additional intervallic root motion schemes that work in concert with the 3^T system. Major triads from $\boldsymbol{Hex}_{(2,3)}$ and their primary dominants form two distinct sets, $\boldsymbol{H}$:=

$\{G^{\Delta}, E\flat^{\Delta}, B^{\Delta}\}$ and $\boldsymbol{J} := \{D^7, B\flat^7, F\sharp^7\}$. With regard to chord/scale determination, we assign Lydian to each major triad in $\boldsymbol{H}$; octatonic and diminished whole tone for each dominant seventh chord in $\boldsymbol{J}$, and draw from the groups $(\boldsymbol{D}_{(n)}, O)$, $(\boldsymbol{Oct}_{(x,y)}, O)$, and $(\boldsymbol{M}_{(x)}, D_8)$.[146]

A group action on $\boldsymbol{H}$ is considered first. $(\boldsymbol{D}_{(2\sharp)}, O)$: $(ab)^{-1}$ produces two orbits of order 3, (143)(256). The second orbit is of interest here. It presents musically as $(F\sharp^{-}, B^{-}, E^{-})$, defined as set $\boldsymbol{Q}^{[1]} := \{F\sharp^{-}, B^{-}, E^{-}\}$. Let us use this permutation over each major triad in $\boldsymbol{H}$, adjusting the pitch level accordingly to produce a unique Lydian scale over each member of $\boldsymbol{H}$. For any x in $\boldsymbol{H}$, there are three available minor triads, built on the 3rd, 6th and 7th of each member of $\boldsymbol{H}$. Therefore, there are three copies of $\boldsymbol{Q}$. We show the relation $\frac{\boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$ as some copy of $\boldsymbol{Q}$ over some member of $\boldsymbol{H}$. Example 38 lists the triads obtained from the group action $(\boldsymbol{D}_{(2\sharp)}, O)$: $(ab)^{-1}$.

Example 38. $\frac{\boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$

$(\boldsymbol{D}_{(2\sharp)}, O)$: $(ab)^{-1}$ $(\boldsymbol{D}_{(2\flat)}, O)$: $(ab)^{-1}$ $(\boldsymbol{D}_{(6\sharp)}, O)$: $(ab)^{-1}$

$$\frac{\boldsymbol{Q}^{[1]}}{x \in \boldsymbol{H}} := \left\{\frac{F\sharp^{-}, B^{-}, E^{-}}{G^{\Delta}}\right\}; \quad \frac{\boldsymbol{Q}^{[2]}}{x \in \boldsymbol{H}} := \left\{\frac{D^{-}, G^{-}, C^{-}}{E\flat^{\Delta}}\right\}; \quad \frac{\boldsymbol{Q}^{[3]}}{x \in \boldsymbol{H}} := \left\{\frac{A\sharp^{-}, D\sharp^{-}, G\sharp^{-}}{B^{\Delta}}\right\}.$$

Members of $\boldsymbol{Q}^{[n]}$ belong to the neo-Riemannian group. Consider $\boldsymbol{Q}^{[1]} := G^{\Delta} \xrightarrow{L} B^{-}$; $G^{\Delta} \xrightarrow{R} E^{-}$. The minor triad built on the 7th above $x \in \boldsymbol{H}$ obtains via conjugation of two neo-Riemannian transformations, $R^L := G^{\Delta} \xrightarrow{L^{-1}} B^{-} \xrightarrow{R} D^{\Delta} \xrightarrow{L} F\sharp^{-}$. Taken as a set union, the three minor triads attain a two-sharp diatonic, equating to G Lydian, the Parent Scale in Russell's

[146] This will impose tensions (9, ♯11) over the major chords and (♭9, ♯9, ♯11, 13) over the dominant seventh chords for octatonic and (♭9, ♯9, ♭5, ♭13) for diminished whole tone.

Lydian Chromatic Approach. Lydian, as a chord scale choice for major triads (and major seventh chords) within tonic systems, is also advocated by Nettles and Graf.[147]

For the dominant harmony, let the set $\boldsymbol{N} := \{\text{major triads} \in \boldsymbol{Oct}_{(x,y)}\}$. We shall work with three copies of $\boldsymbol{N}^{[n]}$. Example 39 lists the triads obtained from group action $(\boldsymbol{Oct}_{(x,y)}, O)\text{: } g$.

Example 39. $\frac{\boldsymbol{N}^{[1]}}{x \in \boldsymbol{J}}$

$$\begin{array}{ccc} (\boldsymbol{Oct}_{(2,3)}, O)\text{: } g & (\boldsymbol{Oct}_{(1,2)}, O)\text{: } g & (\boldsymbol{Oct}_{(0,1)}, O)\text{: } g \\ \frac{\boldsymbol{N}^{[1]}}{x \in \boldsymbol{J}} := \{\frac{\mathrm{D}^{\Delta},\mathrm{F}^{\Delta},\mathrm{A}\flat^{\Delta},B^{\Delta}}{\mathrm{D}^{7}}\}; & \frac{\boldsymbol{N}^{[2]}}{x \in \boldsymbol{J}} := \{\frac{\mathrm{B}\flat^{\Delta},\mathrm{D}\flat^{\Delta},\mathrm{E}^{\Delta},\mathrm{G}^{\Delta}}{\mathrm{B}\flat^{7}}\}; & \frac{\boldsymbol{N}^{[3]}}{x \in \boldsymbol{J}} := \{\frac{\mathrm{F}\sharp^{\Delta},\mathrm{A}^{\Delta},\mathrm{C}^{\Delta},\mathrm{D}\sharp^{\Delta}}{\mathrm{F}\sharp^{7}}\}, \end{array}$$

The choice for invariant major parity is to accentuate the aural experience of a dominant chord's dichroic relationship with its target chord. Mathematically, we define the mapping of a dominant to its target chord as C_2. Note that C_2, the smallest cyclic group has but two members, i, and (12). The (12) exchange explains dominant → target chord motion, as (1 → 2), where 1 = dominant harmony and 2 = target chord harmony. Back relating dominant motion is simply the inverse, represented as (2 → 1), symbolizing target chord → dominant motion.

Toroidal polygons are topologically modified tori that represent polygons with holes.[148] The use of tori has roots in neo-Riemannian theory, wherein the *Tonnetz* forms a discrete lattice on a torus. The torus has a single face, shown as F in Figure 22, two edges, shown as E1 and E2, and vertices that lay at any E1 and E2 intersection. The torus is the product of two orthogonal circles and is therefore cylindrical. There are two distinct paths of motion around a torus: toroidal direction, which is motion about E1, and the poloidal direction, which is motion about E2. A composition of poloidal and toroidal directions produce a diagonal motion across the torus's face.

[147] Nettles and Graf, 164.
[148] Stephen Barr, *Experiments in Topology* (New York, Dover Publications, 1964) 17-8.

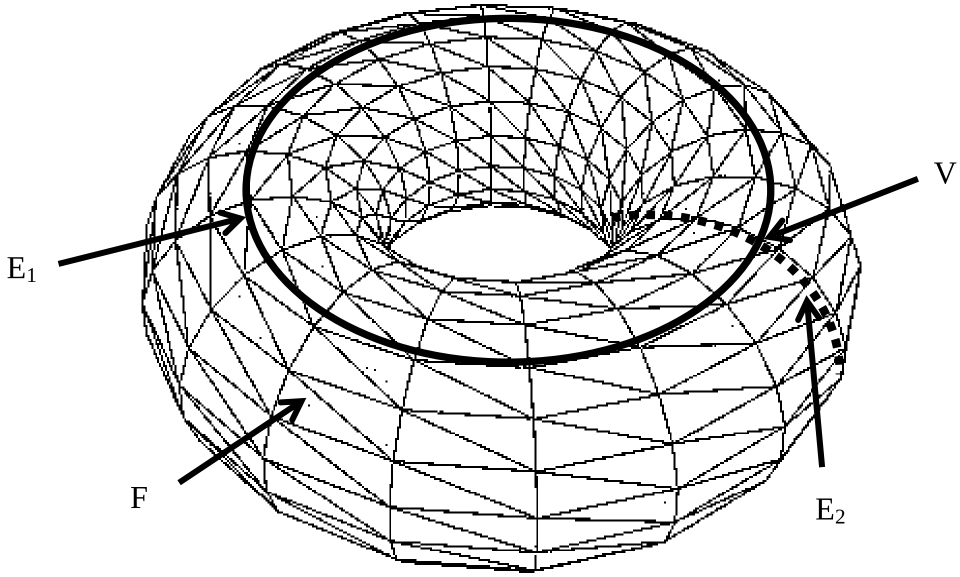


Figure 22. Torus

Thus far, we have studied *n*-gons, shapes in two dimensions and *n*-hedra, certain regular Platonic solids. Regarding regular polyhedra, Leonhard Euler's theorem states,

$$F-E+V = 2.$$

That is, the number of faces less the number of edges plus the number of vertices = 2. This theorem breaks down when applied to tori or doubly connected surfaces such as toroidal polygons; therefore, we modify Euler's theorem, using the equation F−E+V = 0.[149] Agreeing with the modified theorem, the "dominant" toroidal polygon, in the left position of Figure 23 holds 12F – 24E + 12V = 0, and the 3^T toroidal polygon, in the right position holds 9F – 18E + 9V = 0. Figure 23 is a geometric representation of the group,

$$B :=((C_4^3 \rtimes C_3) \times (C_3^3 \rtimes C_3) \times C_2))$$

149 See, B.M.Stewart, *Adventures Among the Toroids: A Study of Orientable Polyhedra with Regular Faces*, 2nd ed. (Okemos, MI.: B.M. Stewart, 1980).

where $C_4^3 \rtimes C_3$ acts on $\frac{\boldsymbol{N}^{[n]}}{x \in \boldsymbol{J}}$ and $C_3^3 \rtimes C_3$ acts on $\frac{\boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$. C_2 generates the dominant-action/3^{T} target chord alternation.

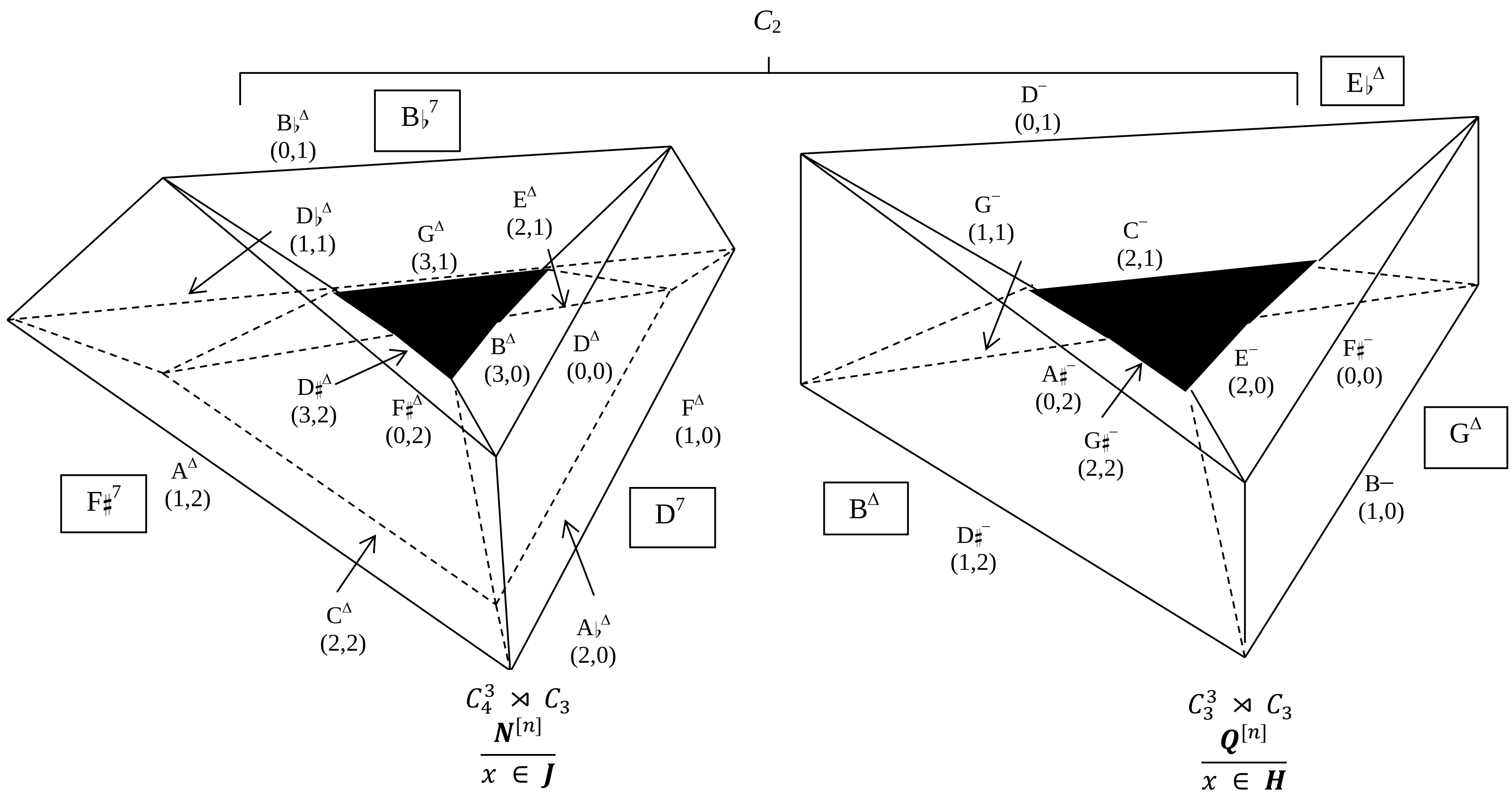


Figure 23. Toroidal polygon

Define the group $W := (\boldsymbol{H}, C_3)$ (stated harmony) and $X := (\boldsymbol{J}, C_3)$ (dominant-action chords). Both of which are generated by counterclockwise toroidal motion about the toroidal polygons. Define the groups $Y := (\boldsymbol{Q}^{[n]}, C_3^3)$ (this includes the neo-Riemannian transformations R, L, R^L) and $Z := (\boldsymbol{N}^{[n]}, C_4^3)$ (triads from $\boldsymbol{Oct}_{(x,y)}$), generated by poloidal motion.

Point coordinates (j,k) show triadic mappings. The first coordinate represents poloidal motion; the second coordinate shows toroidal direction. As the 3^T harmony progresses, it is possible to move from any (j,k) to any other subsequent (j,k) in the adjacent modular space. For example, if we start at F♯$^-$, (0,0), in G$^\Delta$ modular space, and intend to stay in that space, we can move to B$^-$(1,0) or E$^-$(2,0). If we intend to move to E♭$^\Delta$ modular space, the triad following F♯$^-$may be any triad that is contained in E♭$^\Delta$ modular space, D$^-$: (0,1), G$^-$: (1,1), C$^-$: (2,1). As with the torus, the product of toroidal and poloidal direction produces a diagonal motion across the toroidal polygon's face.

We now turn to the presentation of toroidal polygons as Cayley digraphs (a directed graph), which orientate the toroidal polygons on a two-dimensional plane. With the Cayley graph, the intervallic relationships between elements of $\frac{\boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$ and $\frac{\boldsymbol{N}^{[n]}}{x \in \boldsymbol{J}}$ are more easily comprehended.

Definition 23. Cayley diagraph

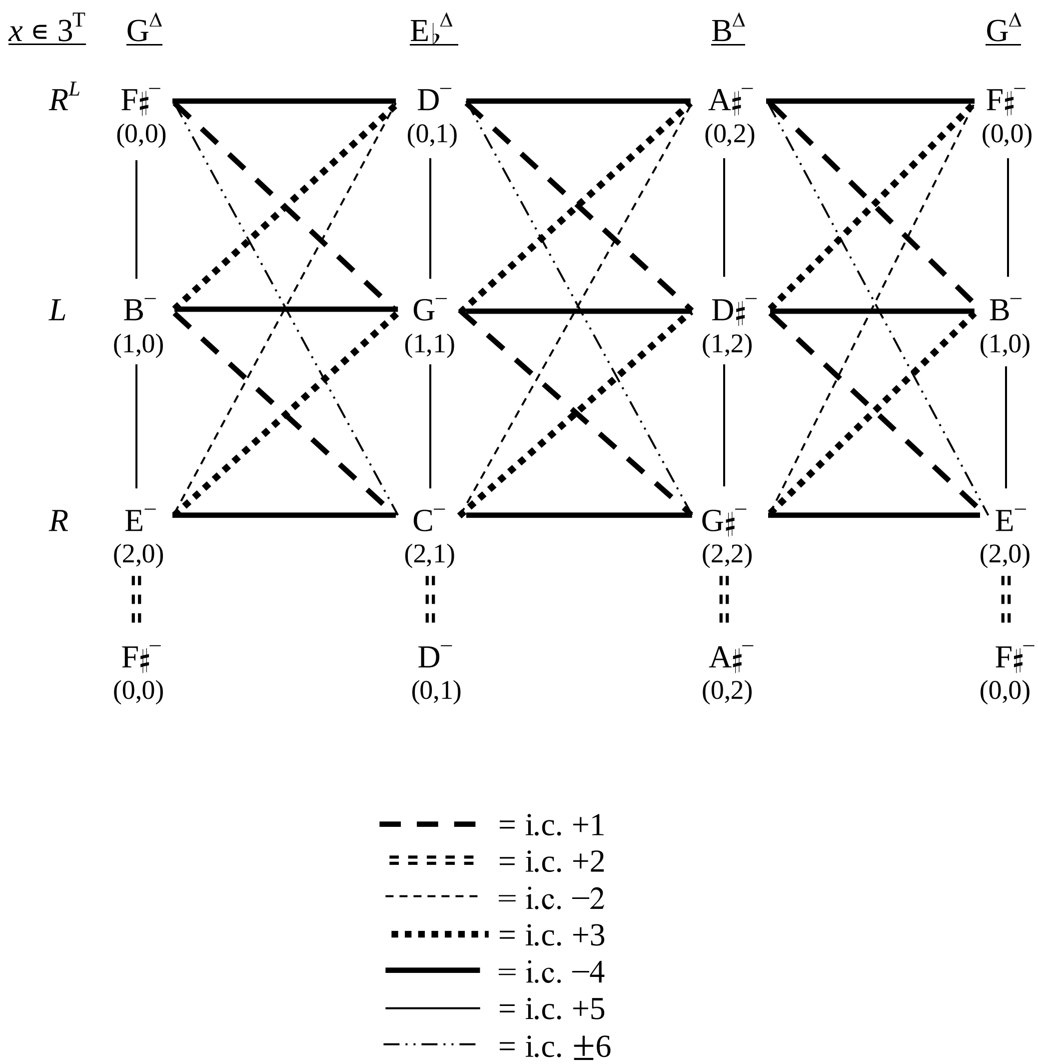
> Let G be a finite group and S a set of generators for G. We define a digraph Cay(S:G), called the Cayley digraph of G with generating set S, as follows.
> 1. Each element of G is a vertex of Cay(S:G).
> 2. For x and y in G, there is arc from x to y if and only if $xs = y$ for some $s \in S$.[150]

In the following Cayley digraphs, stated harmony lays on the x axis, and motion along the x axis is toroidal direction. The superimposed triads lay on the y axis and motion along the y axis

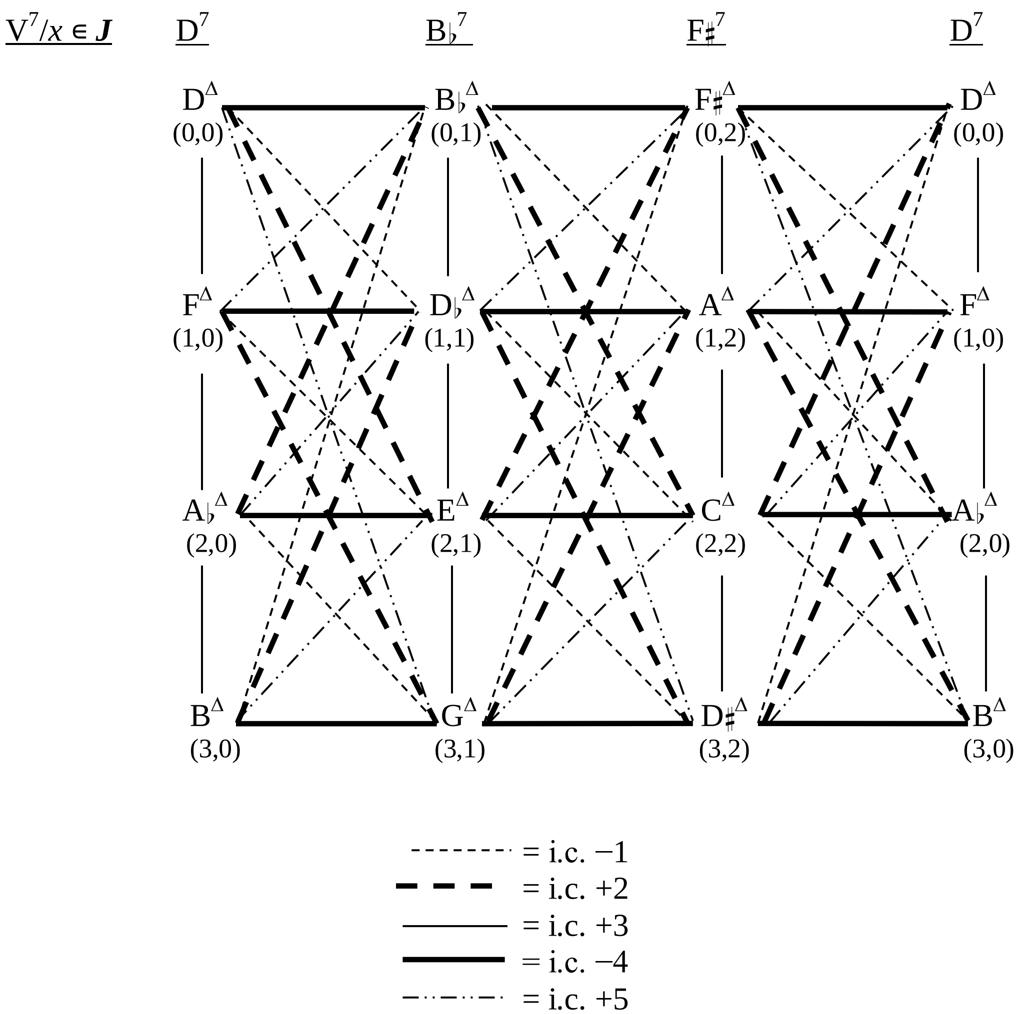
[150] A digraph is a finite set of points, called vertices, and a set of connectors, called arcs, connecting some of the vertices. See Gallian, 497-515.

is poloidal motion. Triads are plotted the digraph's vertices and arcs represent the intervallic relationships, given in directed interval classes below each graph. Remember, the product of a move involving the *x* and *y* axes produce a diagonal motion across the <*x*,*y*> plane.

Example 40.1. $\frac{\boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}} \cong C_3^3 \times C_3$ digraph



Example 40.2. $\frac{\boldsymbol{N}^{[n]}}{x \in \boldsymbol{J}} \cong C_4^3 \times C_3$ digraph[151]



The choice of octatonic and diminished whole tone for the primary dominant-action harmony provides for an altered dominant sonority. The increased chromaticism of an altered

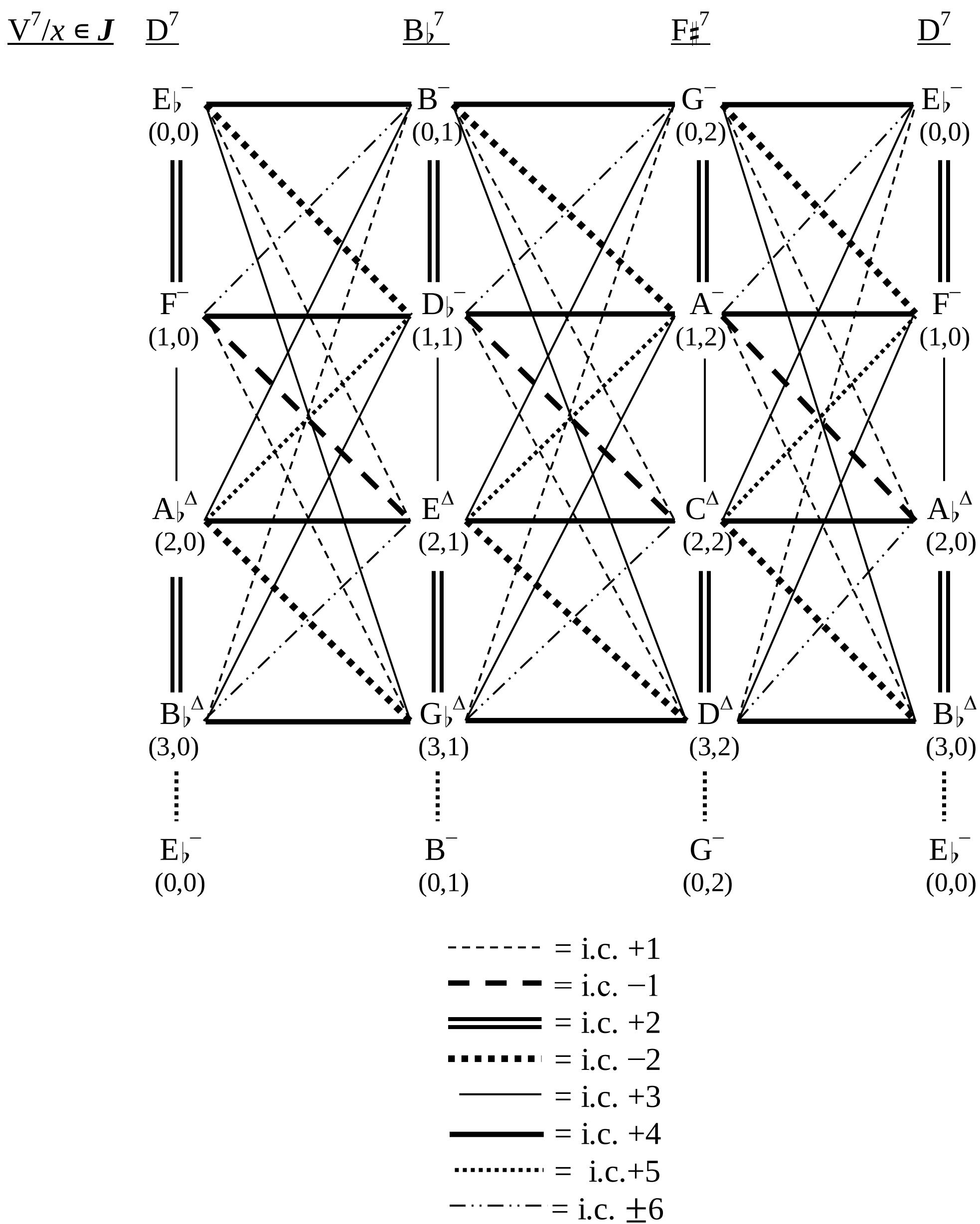
---

[151] This is an example of a Hamiltonian circuit (or Hamiltonian cycle) where the path forms a cycle that visits each vertex exactly once and returns to the start vertex. While a detailed study of Hamiltonian cycles is beyond the scope of this dissertation, they have recently received attention in the literature. See, Giovanni Albini and Samuele Antonini investigates Hamiltonian cycles in the topological dual of the *Tonnetz* (i.e. the successions of triads connected only through *PLR* transformations which visit every minor and major triad only once). See Giovanni Albini and Samuele Antonini, "Hamiltonian Cycles in the Topological Dual of the Tonnetz," in *Mathematics and Computation in Music: Second International Conference, MCM 2009, New Haven, CT., USA, June 19-22, 2009, Proceedings*, eds. Elaine Chew, Adrian Childs, and Ching-Hua Chuan, 1-10 (Berlin: Springer-Verlag, 2009).

primary dominant strengthens the aural relationship between dominant/tonic key areas because the altered tensions (and chord tones) set the dominant further apart from the tonic (or from the intended chord of resolution, whichever the case may be).[152] If we were to provide for an unaltered dominant, the chord/scale choice would be Mixolydian, a modal representation of the diatonic. The choice of an unaltered dominant results in an invariant scale genus in both key areas (dominant and tonic), a representation of the dominant/tonic effect in its most basic form. To further strengthen the dominant/tonic experience through chromaticism, the dominant-action chord/scale choice of diminished whole-tone (seventh mode of the real melodic minor) provides for three altered tensions and one altered chord tone (♭9, ♯9, ♭5, ♭13). Diminished whole tone provides a differing scale genre for the dominant-action chord.

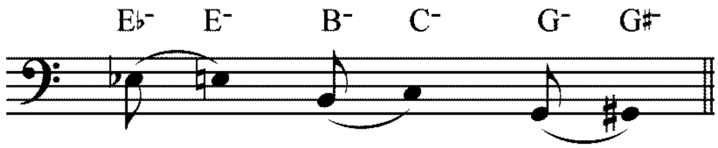
[152] ${}^{sub}V^7$ chords take Lydian ♭7 as a chord scale and have (9, ♯11, 13) as tensions. ${}^{sub}V/^7$ chords are rarely altered beyond the ♯11 because altering the ninth or thirteenth pulls the chord toward the unaltered primary dominant by introducing either a chord tone or unaltered tension belonging to the unaltered dominant.

Example 40.3. $\frac{\boldsymbol{M}_{(x)}}{x \in \boldsymbol{J}} \cong C_4^3 \times C_3$ diagraph



The following examples illustrate musical applications for $B := ((C_4^3 \rtimes C_3) \times (C_3^3 \rtimes C_3) \times C_2))$. The first two examples address a traditional musical technique, that of the neighbor-tone. Example 41.1 builds on the idea of the neighbor-tone by approaching some member of $\boldsymbol{Q}^{[n]}$ by some member of $\boldsymbol{M}_{(x)}$.

Example 41.1. Incomplete lower neighbor motion

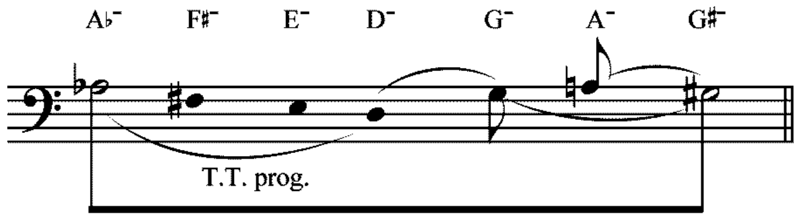


Set	$\frac{z \in \boldsymbol{M}_{(3)}}{x \in \boldsymbol{J}}$	$\frac{R \in \boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$	$\frac{z \in \boldsymbol{M}_{(11)}}{x \in \boldsymbol{J}}$	$\frac{R \in \boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$	$\frac{z \in \boldsymbol{M}_{(7)}}{x \in \boldsymbol{J}}$	$\frac{R \in \boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$
Triad	E♭$^{-}$	E^{-}	B^{-}	C^{-}	G^{-}	G♯$^{-}$
Analysis	V^{7}/G$^{\Delta}$	∊ 3^{T}	V^{7}/E♭$^{\Delta}$	∊ 3^{T}	V^{7}/B$^{\Delta}$	∊ 3^{T}
Stated Harmony	D^{7}	G$^{\Delta}$	B♭^{7}	E♭$^{\Delta}$	F♯^{7}	B$^{\Delta}$

Each member of $\boldsymbol{H}$ contains the neo-Riemannian transformation R; therefore, a reiteration of 3^{T} root motion. The same holds for the dominant chords. Each dominant chord carries a minor triad at T_1, also reiterating 3^{T} root motion.

The next example displays a more complex neighbor-motion, the enclosure around the final triad. This enclosure is preceded by a descending whole-tone descent reminiscent of the whole-tone bass line played during the head of "Giant Steps."

Example 41.2. Whole-tone descent followed by an enclosure around G♯$^{-}$

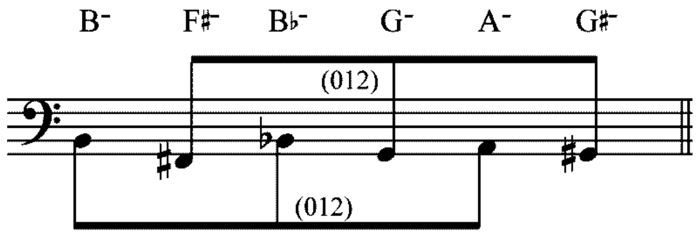


Set	$\frac{z \in \boldsymbol{Oct}_{(2,3)}}{x \in \boldsymbol{J}}$	$\frac{R^{L} \in \boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$	$\frac{z \in \boldsymbol{Oct}_{(\mathbf{1},\mathbf{2})}}{x \in \boldsymbol{J}}$	$\frac{R^{\boldsymbol{L}} \in \boldsymbol{Q}^{[\boldsymbol{n}]}}{x \in \boldsymbol{H}}$	$\frac{z \in \boldsymbol{M}_{(7)}}{x \in \boldsymbol{J}}$	$\frac{z \in \boldsymbol{M}_{(7)}}{x \in \boldsymbol{J}}$	$\frac{R \in \boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$
Triad	A♭$^{-}$	F♯$^{-}$	E^{-}	D^{-}	G^{-}	A^{-}	G♯$^{-}$
Analysis	V^{7}/G$^{\Delta}$	$\in$ 3^{T}	V^{7}/E♭$^{\Delta}$	$\in$ 3^{T}	V^{7}/B$^{\Delta}$		$\in$ 3^{T}
Stated Harmony	D^{7}	G$^{\Delta}$	B♭^{7}	E♭$^{\Delta}$	F♯^{7}		B$^{\Delta}$

Example 42.1 expresses this musically as a set of triads with roots that form two interlocking (012) trichords.[153]. The minor triads (B^{-}, B♭$^{-}$, A^{-}) over members of $\boldsymbol{J}$ obtain from the (j,k) mapping, (3,0) → (0,1) → (1,2). The 3^{T} harmonies use all three neo-Riemannian transformations in $\boldsymbol{Q}$, expressed with the (j,k) mapping, (0,0) → (1,1) → (2,2), which obtains (F♯$^{-}$, G^{-}, G♯$^{-}$) over members of $\boldsymbol{H}$.

[153] Accepting parity equivalence, Example 40.2 also serves to model the minor triads contained in the octatonic, which represents the chord/scale choice for the dominant-action chords. The graph's pitch designations remain the same, parity is changed to minor.

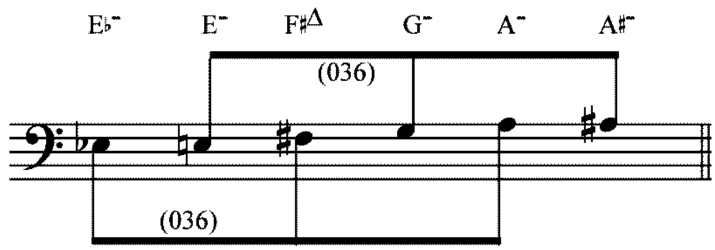
Example 42.1. Interlocking (012) trichords



Set	$\frac{z \in \boldsymbol{Oct}_{(2,3)}}{x \in \boldsymbol{J}}$	$\frac{R^L \in \boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$	$\frac{z \in \boldsymbol{Oct}_{(1,2)}}{x \in \boldsymbol{J}}$	$\frac{L \in \boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$	$\frac{z \in \boldsymbol{Oct}_{(0,1)}}{x \in \boldsymbol{J}}$	$\frac{R \in \boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$
Triad	B⁻	F♯⁻	B♭⁻	G⁻	A⁻	G♯⁻
Analysis	V^7/G^Δ	∊ 3^T	V^7/E♭$^\Delta$	∊ 3^T	V^7/B^Δ	∊ 3^T
Stated Harmony	D^7	G^Δ	B♭^{7}	E♭$^\Delta$	F♯^{7}	B^Δ

Example 41.2 displays triads with roots forming interlocking (036) trichords. The neighbor-action remains in the choice of sets among members of H and J. The background permutational scheme relates each $\boldsymbol{M}_{(n)}$ generated set over $\boldsymbol{J}$ by $n \cdot 4$. Regarding the neo-Riemannian transformations $\boldsymbol{Q}^{[n]}$ over members of $\boldsymbol{H}$: R, is followed by L, which closes on the conjugation R^L.

Example 41.2. Interlocked (036) trichords



Set	$\frac{z \in \boldsymbol{M}_{(3)}}{x \in \boldsymbol{J}}$	$\frac{R \in \boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$	$\frac{z \in \boldsymbol{M}_{(11)}}{x \in \boldsymbol{J}}$	$\frac{L \in \boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$	$\frac{z \in \boldsymbol{M}_{(7)}}{x \in \boldsymbol{J}}$	$\frac{R^{L} \in \boldsymbol{Q}^{[n]}}{x \in \boldsymbol{H}}$
Triad	$E\flat^{-}$	E^{-}	$F\sharp^{\Delta}$	G^{-}	A^{-}	$A\sharp^{-}$
Analysis	V^{7}/G^{Δ}	$\in 3^{T}$	$V^{7}/E\flat^{\Delta}$	$\in 3^{T}$	V^{7}/B^{Δ}	$\in 3^{T}$
Stated Harmony	D^{7}	G^{Δ}	$B\flat^{7}$	$E\flat^{\Delta}$	$F\sharp^{7}$	B^{Δ}

CHAPTER 5. CONCLUSIONS AND ADDITIONAL RESEARCH

5.1. Conclusions

The Permutational Triadic Approach offers a syntactic theory to address non-traditional uses of triads. The theory acts as descriptively: it incorporates permutation group theory to organize triads, and represents the resulting group actions geometrically. These representations demonstrate abstract algebraic structures on a familiar geometric objects, offering the musician a tool by which to conceptualize underlying mathematical groups. In other words, it provides the musician with a visual template. The number of specific geometric elements in the shape, e.g., vertices, edges, faces, helps determine what groups can model a particular set of musical objects, based on the set's cardinality. From the descriptive view, the theory offers an analytical technique encompassing functional harmony and the chord/scale relationship in a manner previously unexplored in jazz theory. We can now organize triadic permutations according to the group actions that generate them, and identify other triadic permutations that relate under group actions. These relations include permutations that belong to the same subgroup, that share an equivalence relation (for example, conjugacy class), or are members of a group product.

The theory is also prescriptive, having compositional and improvisational uses. The prescriptive aspect also carries with it new pedagogical implications wherein the systematic approach to triadic permutation acts as a medium through which teachers, students, and fellow musicians can communicate.

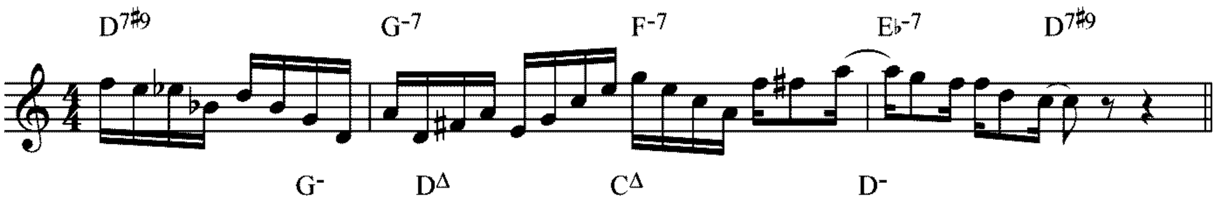
5.2. Additional Musical Applications

Many of the above mathematical concepts could serve as a topic for additional study. Any of the various mathematical groups might form the basis for an investigation into the repertoire; topics such as cyclic groups in jazz, dihedral groups in post-Coltraneian harmony, or occurrences of triadic permutations based on the alternating group acting as a subgroup of the octahedral symmetry group in tonic systems come to mind. Direct products and semidirect products were a reoccurring theme in many of the analyses. As such, it is possible to undertake a study limited to the identification and explanation of these and other group products in jazz compositions, arrangements, and improvisations.

Set composites are another topic that warrants further investigation. Set composites are redefined sets where a permutation group models triads from more than one scale genre. For example, consider the altered V^7 in the key of C major, $G^{7\ alt}$. $\boldsymbol{Oct}_{(1,2)}$ and G diminished whole-tone are two possible chord/scale partners that generate the triad pairs $\{(B\flat^{\Delta}, E^{\Delta}) \in \boldsymbol{Oct}_{(1,2)}, (D\flat^{\Delta}, E\flat^{\Delta}) \in \boldsymbol{M}_{(8)}\}$. The set union of the four triads attains a scale-like collection {1,2,3,4,5,7,8,10,11}, which is not in the scale roster. Define the set containing these four triads and model that set on C_4, D_8, V_4, and A_4 for use over $G^{7\ alt}$. Likewise, $C^{\Delta 7\sharp 5}$ takes Lydian augmented and $\boldsymbol{Hex}_{(3,4)}$ as two possible chord/scale partners. From these two scales define the set $\{(B^{-}, A^{-}) \in \boldsymbol{M}_{(9)}, (A\flat^{-}, E^{\Delta}) \in \boldsymbol{Hex}_{(3,4)}\}$, whose set union attains {0,2,3,4,6,8,9}, also a scale-like collection not found in the scale roster. For example, in the case of $C^{\Delta 7\sharp 5}$, take the cyclic permutation $(A\flat^{-}, A^{-}, B^{-}, E^{\Delta})$, where $A\flat^{-}$ acts a lower neighbor to A^{-}, and B^{-} acts a modally inflected dominant of E^{Δ}.

Set generators in this study were limited to the collections listed in the scale roster. This fact does not mean that the blues scale and the be-bop scales cannot serve as set generators. They can, in fact, as was touched upon in the Charlie Parker Example 1.13. Michael Brecker uses a similar technique in his solo on "El Niño," [154] wherein he plays (G^-, D^Δ, C^Δ, D^-). This set union produces the scale collection (G, A, B♭, C, D, E, F, F♯), which is either a Dorian bebop scale, or a blues scale influenced dominant bebop scale through the inclusion of the blue note $\flat\hat{3}$, depending on the reading.

Example 43. "El Niño," excerpt



A final word regarding set definition and set cardinality expansion: in regard to the 3^T chords (G^Δ, $E\flat^\Delta$, B^Δ), the three minor triads derived from a single permutation discussed in the 3^T revisited section, each of these triads could be expanded into minor pentatonic scales without introducing avoid tones. Moreover, the minor pentatonic scales could be embellished by adding $\flat\hat{5}$, thus turning them into blues scales. Further study into the expansion of triads obtained by group actions into pentatonic collections that are based on the fundamental triadic structure is worthwhile.

[154] Michael Brecker, *The Michael Brecker Collection*, transcribed by Carl Coan (Milwaukee: Hal Leonard, n.d.), 53.

5.3. Additional Mathematical Questions

This dissertation focused on 2-dimentional polygons and platonic solids in 3-dimensional Euclidian space to model rotational and dihedral groups. There is a wealth of other groups that can model the sets defined here. One of which is the symmetry group of the Fano plane, the smallest projective plane to address sets of order seven.

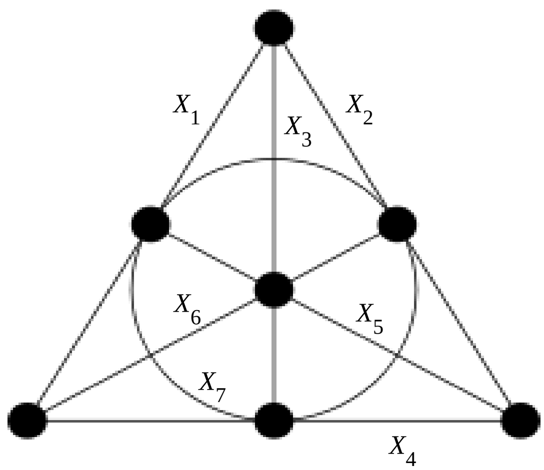


Figure 24. Fano plane

The Fano plane has been studied in musical contexts by others, such as Carlton Gamer and Robin Wilson, who use it to investigate geometric duality and the division of the octave into units other than twelve to facilitate microtonal compositions.[155] Their work offers an analysis of Gamer's *Fanovar,* a composition governed by the Fano plane's structure (but not its symmetries). David Lewin demonstrates how aspects of the Fano plane's geometry can be

[155] Carlton Gamer and Robin Wilson, "Microtones and Projective Planes," in *Music and Mathematics: From Pythagoras to Fractals,* ed. John Fauvel, Raymond Flood and Robin Wilson, 149-61(Oxford: Oxford University Press, 2003).

projected compositionally over a cantus firmus.[156] In the course of his discussion, Lewin describes group generators for the Fano plane's symmetry group, and offers numerous group actions as components of musical examples. Given that it is possible to model a set of degree 7 on the Fano plane's seven elements, the Fano plane's symmetry group can help ameliorate the order-7 *p*-group limitation to cyclic groups.[157]

Robert Peck, in an unpublished response to Jack Douthett's "Filtered Point-Symmetry and Dynamical Voice-Leading,"[158] states, "each of the seven projective lines in the Fano plane may model the [non-commutative] multiplication scheme of Hamilton's quaternions…the order eight group of the quaternions [written Q_8]," and provides a coinciding operation to David Lewin's Q relations.[159]

Example 44. Lewin's Q3 operation on $\boldsymbol{S} := \boldsymbol{Oct}_{(0,1)}$[160]

$$\boldsymbol{S} := \{0,1,3,4,6,7,9,10\}$$

$$Q_3 := (0,3,6,9)(1,10,7,4)$$

[156] David Lewin, "Some Compositional Uses of Projective Geometry," *Perspectives of New Music* 42, no.2 (2004): 12-63.

[157] The symmetry group of the Fano plane is of order 168. One way to reduce the number of group actions to be considered is to use subgroups. It would be beneficial to derive the subgroups from the Fano plane symmetry group using a group structure already introduced in this document. There exist two subgroups of the Fano plane group that are isomorphic to the group *O*. Further study entails investigating these subgroups, as well as the other subgroups that are conjugate to them in the Fano plane group.

[158] Jack Douthett, "Filtered Point-Symmetry and Dynamical Voice-Leading," in *Music Theory and Mathematics: Chords, Collections and Transformations*, ed. Jack Douthett, Martha M. Hyde, Charles J. Smith, 72-106 (Rochester: University of Rochester Press, 2008).

[159] Robert Peck, "A Response to Jack Douthett's 'Filtered Point-Symmetry and Dynamical Voice-Leading'" (conference on Musical Systems in Memory of John Clough, University of Chicago, Chicago, IL, July 8, 2005): 2-4. Peck further develops this concept in "Imaginary Transformations." *Journal of Mathematics and Music* 4, no. 3 (2010): 157-71.

[160] David Lewin, *Generalized Musical Intervals and Transformations*, (Oxford: Oxford University Press, 2007): 251-3.

Redefine $\boldsymbol{S}$ = pitch classes in $\mathrm{Oct}_{(0,1)}$ as $\boldsymbol{T}$ = the consonant triads in $\mathrm{Oct}_{(0,1),}$

x ∈ S	y ∈ T
0	1 := ({0,4,7} = C^{Δ})
1	2 := ({0,3,7} = C^{-})
3	3 := ({3,7,10} = $\mathrm{E}\flat^{\Delta}$)
4	4 := ({3,6,10} = $\mathrm{E}\flat^{-}$)
6	5 := ({6,10,1} = $\mathrm{F}\sharp^{\Delta}$)
7	6 := ({6,9,1} = $\mathrm{F}\sharp^{-}$)
9	7 := ({9,1,4} = A^{Δ})
10	8 := ({9,0,1} = A^{-})

Q_3 acting on $\boldsymbol{Oct}_{(0,1)}$ produces (C^{Δ}, $\mathrm{E}\flat^{\Delta}$, $\mathrm{F}\sharp^{\Delta}$, A^{Δ})(C^{-}, A^{-}, $\mathrm{F}\sharp^{-}$, $\mathrm{E}\flat^{-}$), in cyclic notation using the numeration from $\boldsymbol{T}$, which corresponds to the integer assignment used previously in the section on octahedral symmetry, (1,3,5,7)(2,8,6,4). This is a permutation that certainly exists in S_8, but is not found in groups D_{16}, O, or Oh that were covered in this study. Instead, the permutation is a member of ($\boldsymbol{Oct}_{(0,1)}$, Q_8); therefore, Q_8 offers another avenue to investigate permutations on the set of consonant triads contained in an octatonic collection.[161]

Garzone's Triadic Chromatic Approach allows for any triad to follow any other triad regardless of quality, meaning consonant and dissonant triads are available. Campbell and Weiskopf also employ consonant and dissonant triads in their work. A full triadic representation of any scale in the roster presented in this document would of course need to include dissonant triads. To apply such expanded set definitions, take the set's cardinality to find the appropriate geometric model to identify the corresponding group structure. In the case of the diatonic (of degree 7), this determination means that the geometric model must change to accommodate a

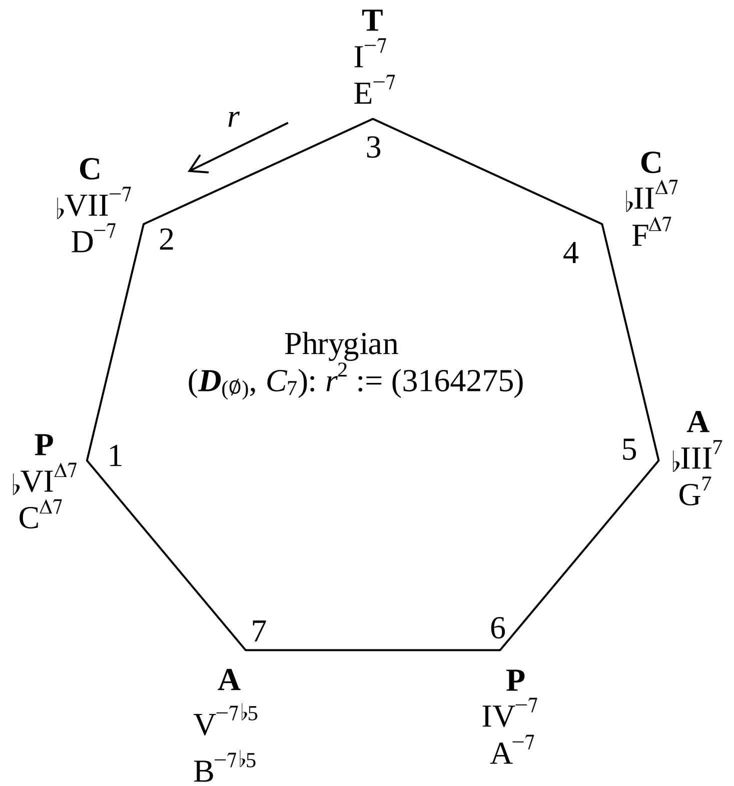
[161] The octahedral group acting on the eight consonant triads in $\mathrm{Oct}_{(x,y)}$ has a subgroup of order eight that contains members of the T/I group. Members of this subgroup commute with eight members of S_8: $\{i \Leftrightarrow i\}$,
$\{g := (1,3,5,7)(2,4,6,8) \Leftrightarrow Q_3 := (1,3,5,7)(2,8,6,4)\}$, $\{g^2 := (15)(26)(37)(48) \Leftrightarrow Q_6 := (15)(26)(37)(48)\}$,
$\{g^{-1} := (1753)(2864) \Leftrightarrow Q_9 := (1753)(2468)\}$, $\{g^{-1}h^2 := (12)(38)(47)(56) \Leftrightarrow X_1 := (12)(38)(47)(56)\}$,
$\{(h^g)^2 := (14)(23)(58)(67) \Leftrightarrow X_3 := (14)(23)(58)(67)\}$, $\{gh^2 := (16)(25)(34)(78) \Leftrightarrow X_5 := (16)(25)(34)(78)\}$,
$\{h^2 := (18)(27)(36)(45) \Leftrightarrow X_7 := (18)(27)(36)(45)\}$.

degree 7 p-group, allowing C_7 and D_{14} (or alternatively, the Fano plane) as group possibilities. The hexatonic collection is not required to shift to a different geometric model for full triadic representation. This symmetric scale contains six consonant triads and two augmented triads, totaling eight triads. The octahedron was previously used to model the hexatonic, wherein triads were placed on vertices. Given that the octahedron has six vertices, twelve edges and eight faces, we need not leave the octahedron. Instead, by invoking geometric duality (with the cube), we can model the full set of triads on the octagon's eight faces. This does not hold for the octatonic, which contains eight consonant triads and eight dissonant triads, totaling sixteen triads. To model a set of degree 16, the hyper-cube or tesseract is a possibility. The hyper-cube, which has sixteen vertices, is isomorphic to the symmetry group $O \times C_2$ in four dimensions.

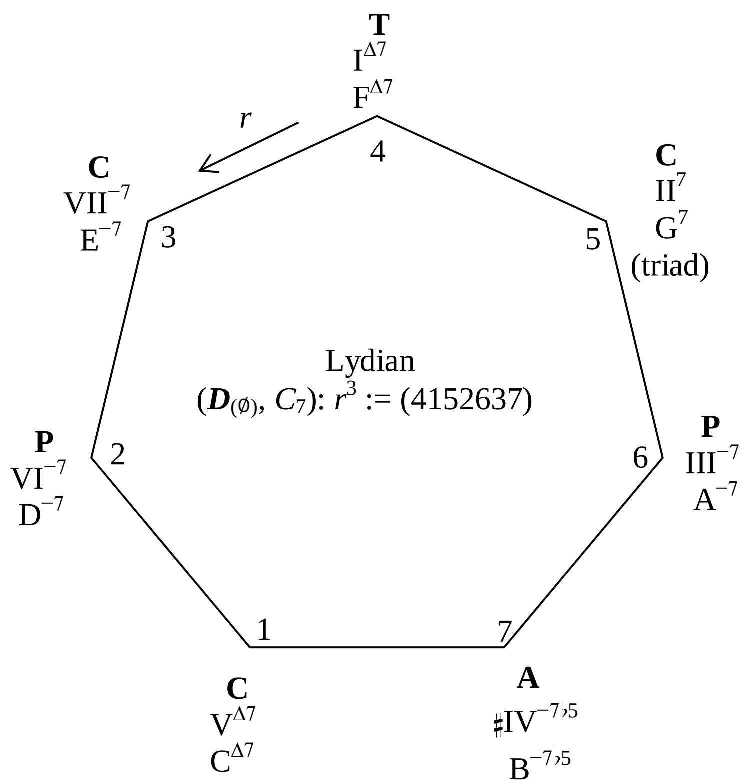
The Permutational Triadic Approach uses permutation group theory to explain what is well-known to the jazz musician; it offers no new jazz theory. It uses mathematics that is well-known to specialists in the mathematics and music subdiscipline of music theory; it offers no new mathematics. Its beauty lies in its synergetic application that acts as an invitation to those working in both disciplines to join in a conversation that will surely benefit all who choose to participate.

A.1. Diatonic Modes

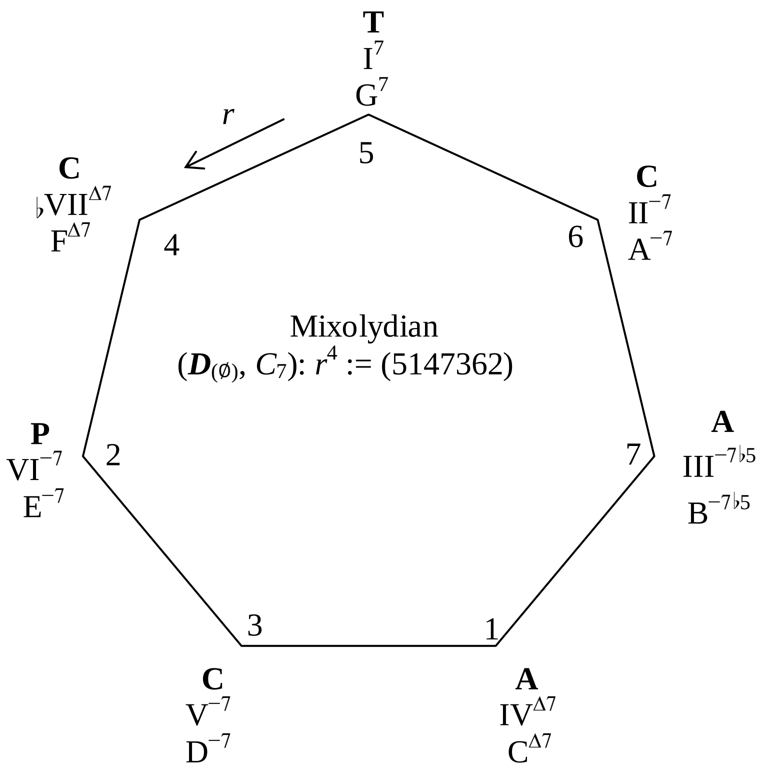
A.1.1. Phrygian



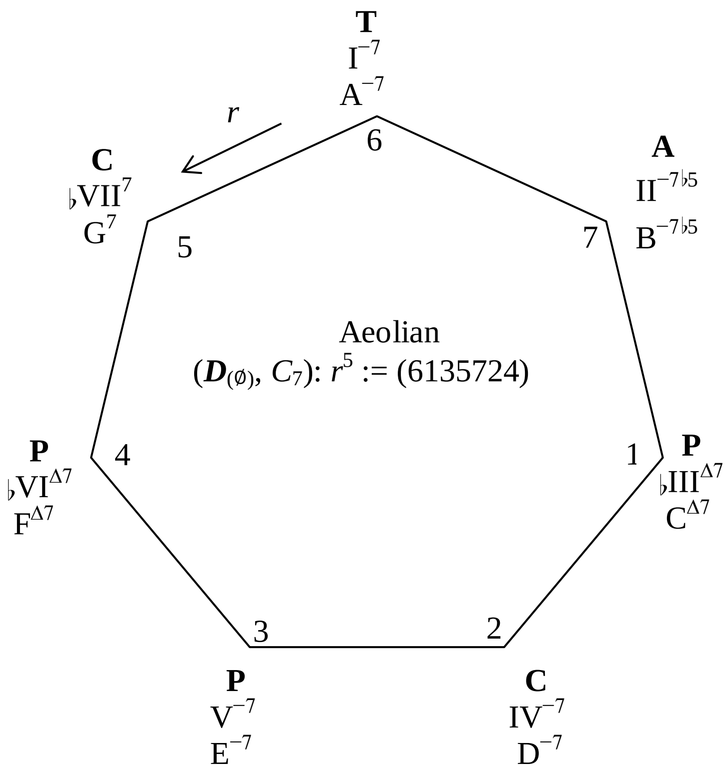
A.1.2. Lydian


T
$I^{\Delta7}$
$F^{\Delta7}$
r
4
C
VII^{-7}
E^{-7}
3
C
II^7
G^7
(triad)
5
Lydian
$(\boldsymbol{D}_{(\emptyset)}, C_7)$: $r^3 := (4152637)$
P
VI^{-7}
D^{-7}
2
P
III^{-7}
A^{-7}
6
1
7
C
$V^{\Delta7}$
$C^{\Delta7}$
A
$\sharp IV^{-7\flat5}$
$B^{-7\flat5}$

A.1.3. Mixolydian



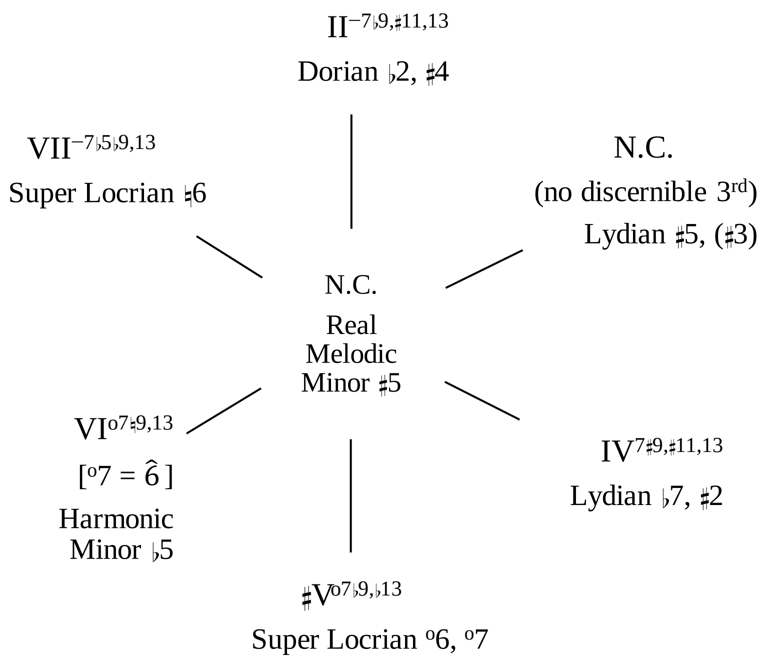
A.1.4. Aeolian



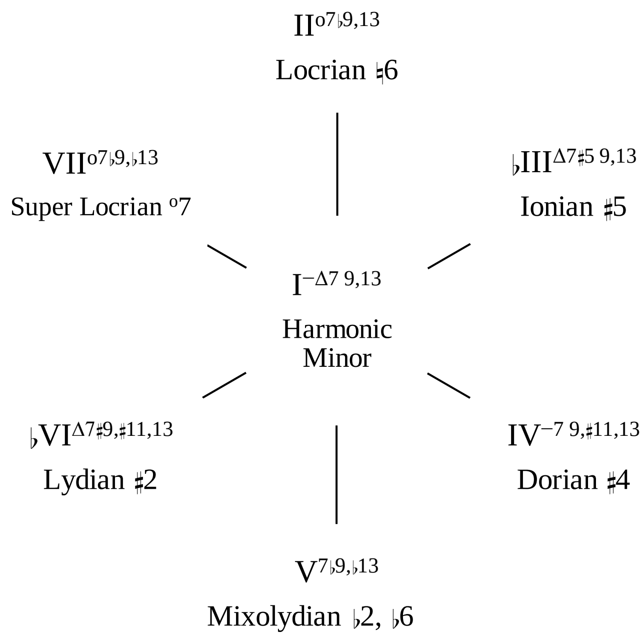
T
I^{-7}
A^{-7}
r
6
A
$II^{-7\flat5}$
$B^{-7\flat5}$
7
C
$\flat VII^{7}$
G^{7}
5
Aeolian
$(\boldsymbol{D}_{(\emptyset)}, C_7)$: r^5 := (6135724)
P
$\flat VI^{\Delta7}$
$F^{\Delta7}$
4
P
$\flat III^{\Delta7}$
$C^{\Delta7}$
1
3
2
P
V^{-7}
E^{-7}
C
IV^{-7}
D^{-7}

A.2. Synthetic Scale Modes

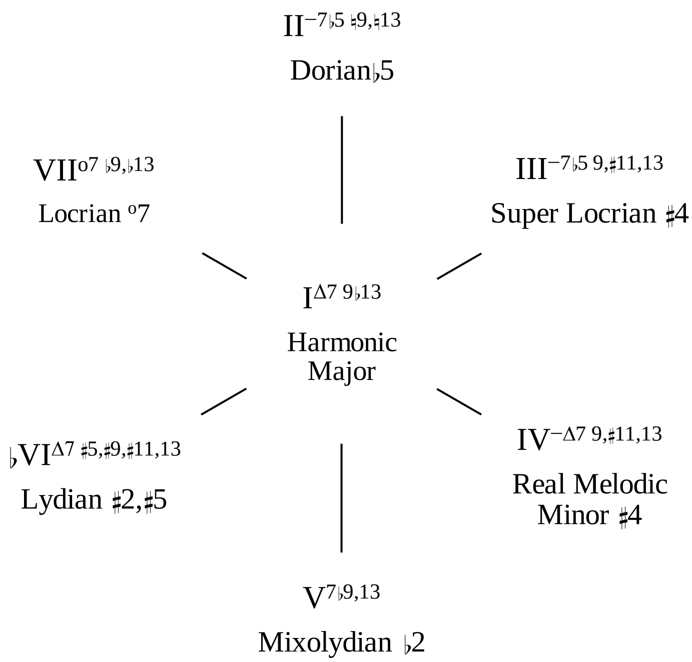
A.2.1. Real Melodic Minor ♯5



A.2.2. Harmonic Minor



A.2.3. Harmonic Major



A.2.4. Double Harmonic

♭II$^{7\ \sharp 9,\sharp 11}$
[♭7 = $\hat{6}$]
Lydian ♯2,♯6 or
⊂ (dominant)
Be-bop ♯2,♯4

VII$^{o7\ \flat 9[\natural 9],\flat 13}$
Locrian º3,º7

III$^{o7\ \flat 9,\flat 13}$
Phrygian º4,º7

I$^{\Delta 7\flat 9,\flat 13}$
Double
Harmonic

♭VI$^{\Delta 7\sharp 5\ 9,\sharp 11,13}$
Ionian ♯2,♯5

IV$^{-\Delta 7\ 9,\sharp 11,\flat 13}$
Harmonic
Minor ♯4

V$^{7\flat 5\ \flat 9,13}$
Mixolydian
♭2,♭5

A.2.5. Double Harmonic ♯5

II$^{\Delta 7\ \sharp 9}$
Lydian ♯2,♯6
or [♭7 = $\hat{6}$]
II$^{7\sharp 5,\sharp 9}$
⊂ (dominant)
Be-bop ♯2, ♯4

N.C.
(no discernible 3rd)
No discernible scale

III$^{\Delta 6\flat 9}$
⊂ (major)
Be-bop ♭2

I$^{\Delta 7\sharp 5\ \flat 9,13}$
Double
Harmonic ♯5

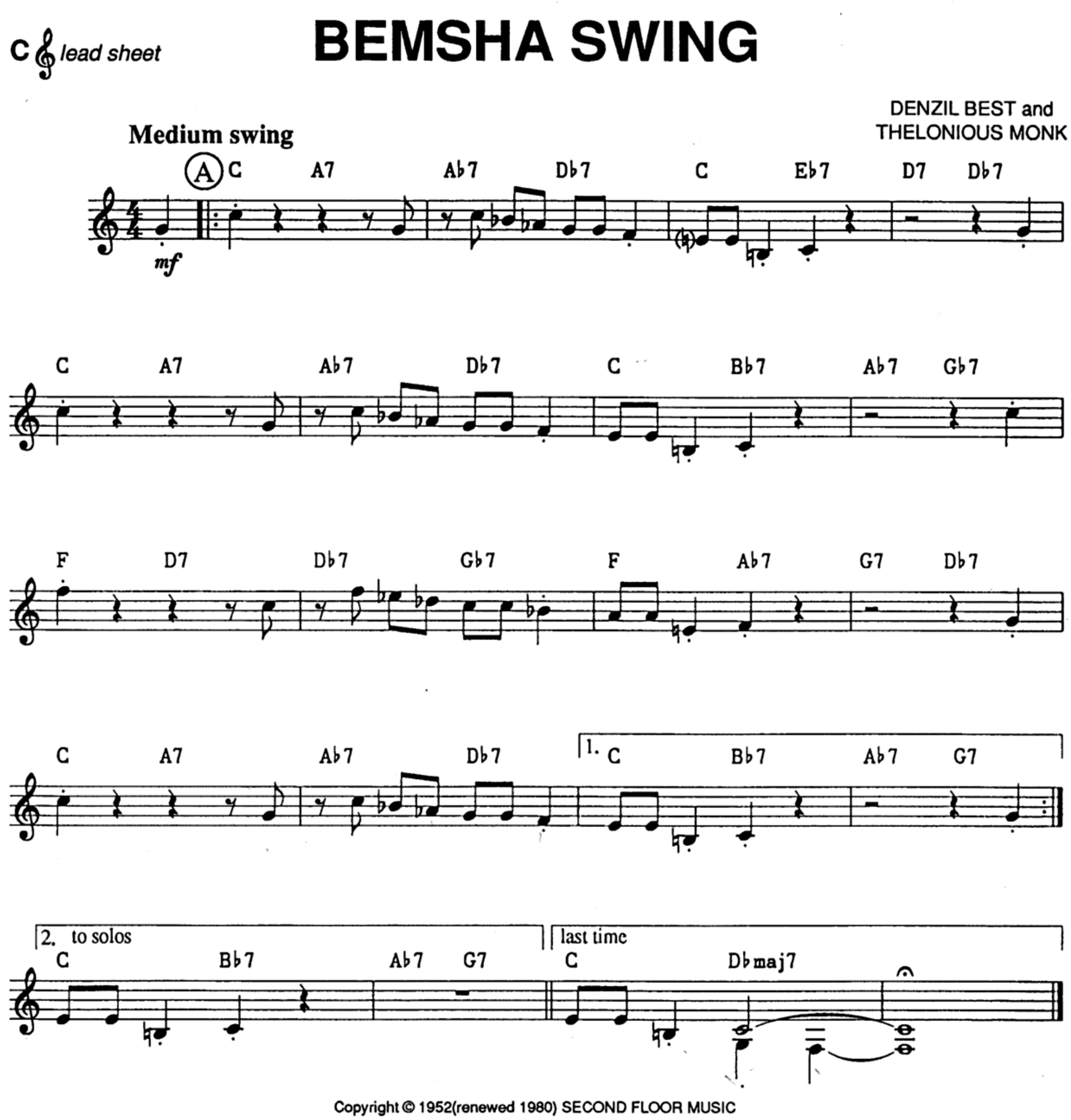
VI$^{-\Delta 7\ 9,\flat 13}$
Harmonic Minor
º4
or
[3 = $\hat{4}$] VI$^{\Delta 7\ 9,\flat 13}$
Harmonic Major
♮2, ♭2 (no $\hat{4}$)

IV$^{\Delta 7\ \sharp 9,\sharp 11,\flat 13}$
Lydian ♯2,♭6

♯V$^{\Delta 6\sharp 5\ \flat 9,\sharp 9}$
[3 = $\hat{4}$, 6 = $\hat{7}$]
⊂ Ionian ♭2, ♯2, ♯5

B.1. "The Beatles"

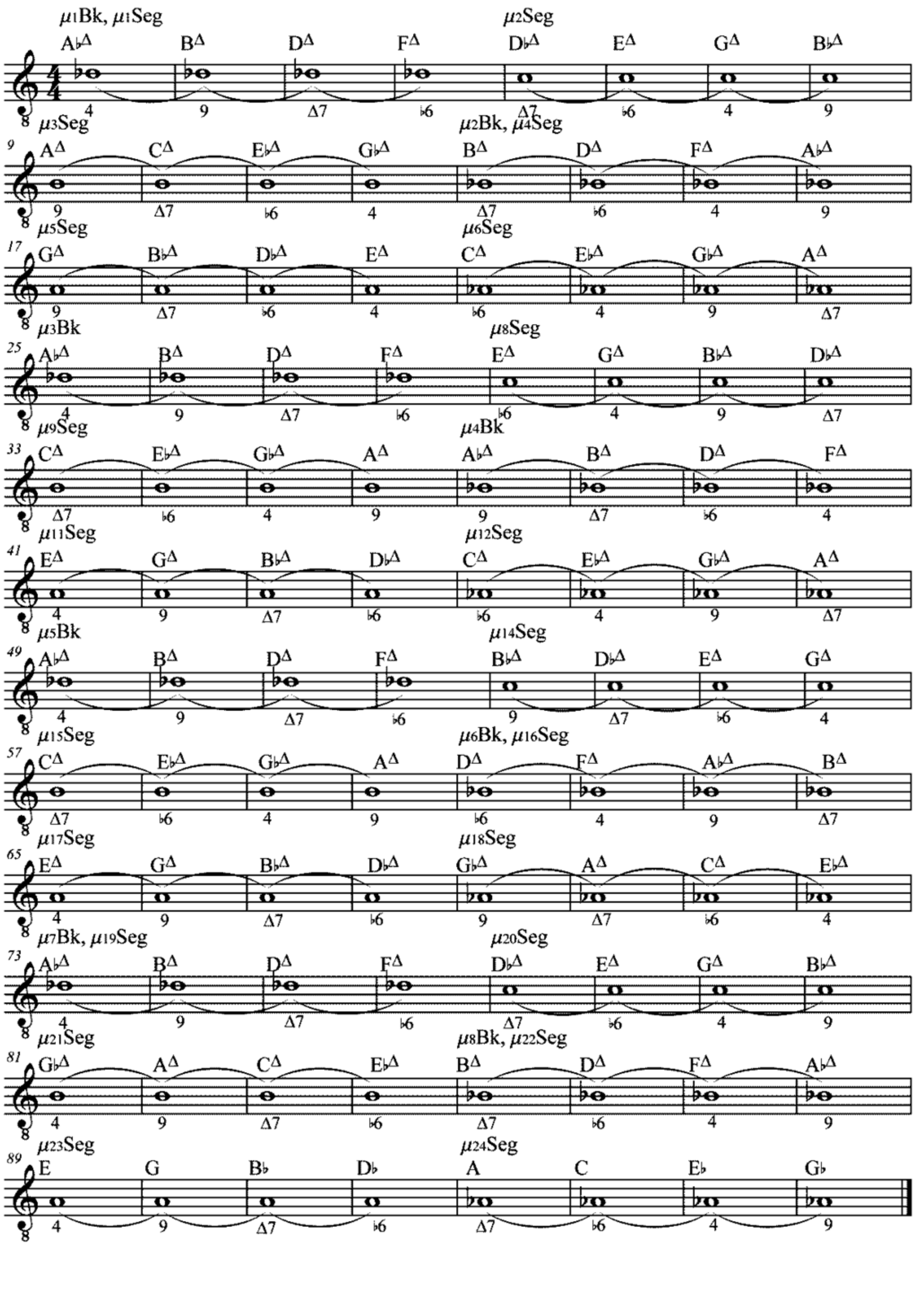
B.2. "Bemsha Swing"[162]



[162] The analysis is based on a transcription by Rick Peckham, received October 1987. The publisher of "Bemsha Swing" provided a lead sheet that is contained in Appendix B.2. There are two minor discrepancies. The publisher's lead sheet has C^{Δ}—$B\flat^{7}$—$A\flat^{7}$—G^{7} in mm. 15-6. $A\flat^{7}$ and G^{7} are tritone substitutions of the chords in the transcription, and the $B\flat^{7}$ is analyzed as a member of the permutation in mm. 5-6 of the Peckham transcription. The final chord on the publisher's lead sheet is $D\flat^{\Delta 7}$. My hearing of the structural final chord agrees with Peckham, although I do hear a second inversion $D\flat^{\Delta}$ as a suffix embellishment to the arrival of the structural $A\flat^{\Delta}$.

B.3. "The Father and the Son and the Holy Ghost"

B.4. “Hotel Vamp”



B.5. "Ma Belle Hélène"

“Ma Belle Hélène,” p.2

APPENDIX C. AEBERSOLD/BAKER SCALE SYLLABUS

Chord/Scale Symbol	Scale Name	Scale ($\hat{1}$=C)
1. Five Basic Categories		
C	Major	(C,D,E,F,G,A,B)
C^7	Mixolydian	(C,D,E,F,G,A,B♭)
C^-	Dorian	(C,D,E♭,F,G,A,B♭)
$C^{-7\flat5}$	Locrian	(C,D♭,E♭,F,G,A♭,B♭)
C°	Diminished$_{(W,H)}$ [Octatonic$_{(2,3)}$]	(C,D,E♭,F,G♭,A♭,B♭,B)
2. Major Scale Choices		
C^{Δ}	Major	(C,D,E,F,G,A,B)
C	Major Pentatonic	(C,D,E,G,A)
$C^{\Delta\sharp4}$	Lydian	(C,D,E,F♯,G,A,B)
C^{Δ}	Bebop (Major)	(C,D,E,F,G,A♭,A,B)
$C^{\Delta\flat6}$	Harmonic Major	(C,D,E,F,G,A♭,B)
$C^{\Delta\sharp5,\sharp11}$	Lydian Augmented	(C,D,E,F♯,G♯,A,B)
C	Augmented [Hexatonic$_{(11,0)}$]	(C,D♯,E,G,A♭,B)
C	6th Mode of Harmonic Major	
C	Diminished$_{(H,W)}$ [Octatonic$_{(0,1)}$]	(C,D♭,E,E♭,F♯,G,A,B♭)
C	Blues Scale	(C,E♭,F,G♭,G,B♭)
3. Dominant Seventh Scale Choices		
C^7	Mixolydian	(C,D,E,F,G,A,B♭)
C^7	Major Pentatonic	(C,D,E,G,A)
C^7	Bebop (Dominant)	(C,D,E,F,G,A,B♭,B)
$C^{7\ \flat9}$	Spanish/Jewish [Mixolydian ♭2, ♭6] (5th Mode of Harmonic Minor)	(C,D♭,E,F,G,A♭,B♭)
$C^{7\ \sharp11}$	Lydian Dominant (4th Mode of Melodic Minor)	(C,D,E,F♯,G,A,B♭)
$C^{7\ \flat6}$	Hindu [Mixolydian ♭6] (5th Mode of Melodic Minor)	
$C^{7aug(\#5,\#11)}$	Whole Tone	(C,D,E,F♯,G♯,B♭)
$C^{7\ \flat9(\sharp9,\sharp11)}$	Octatonic$_{(0,1)}$	(C,D♭,E,E♭,F♯,G,A,B♭)
$C^{7\ \#9\ (\#5,\flat9,\#11)}$	Diminished Whole Tone [Altered Scale, Super Locrian]. (7th Mode of Melodic Minor)	(C,D♭,E♭,F,G,A♭,B♭)
C^7	Blues Scale	(C,E♭,F,G♭,G,B♭)
Dominant 7th sus4	Can also be seen as G^-/C	

C^{7sus4}	Mixolydian	(C,D,E,F,G,A,B)
C^{7sus4}	Major Pentatonic built on ♭7	(C,D,F,G,B♭)
C^{7sus4}	Bebop [Dominant]	(C,D,E,F,G,A,B♭,B)
4. Minor Scale Choices		
C^{-} or C^{-7}	Dorian	(C,D,E♭,F,G,A,B♭)
C^{-} or C^{-7}	Minor Pentatonic	(C,E♭,F,G,B♭)
C^{-} or C^{-7}	Bebop (Minor)	(C,D,E♭,E,F,G,A,B♭)
$C^{-\Delta7}$	Real Melodic Minor	(C,D,E♭,F,G,A,B)
C^{-}, C^{-6}, $C^{-\Delta7}$	Bebop (Minor no.2)	(C,D,E♭,F,G,A♭,A,B)
C^{-}, C^{-7}	Blues Scale	(C,D,E♭,F,G♭,G,B♭)
$C^{-\Delta\flat6}$	Harmonic Minor	(C,D,E♭,F,G,A♭,B)
C^{-}, C^{-7}	Diminished $_{(W,H)}$ [Octatonic $_{(2,3)}$]	(C,D,E♭,F,G♭,A♭,A,B)
C^{-}, $C^{-}(\flat2, \flat6)$	Phrygian	(C,D♭,E♭,F,G,A♭,B♭)
C^{-}, C^{-} $(\flat6)$	Aeolian	(C,D,E♭,F,G,A♭,B♭)
5. Minor Seventh Flat Five Scale Choices		
$C^{-7\flat5}$	Locrian	(C,D♭,E♭,F,G♭,A♭,B♭)
$C^{-7\flat5\ \natural9}$	Locrian ♯2	(C,D,E♭,F,G♭,A♭,B♭)
$C^{-7\flat5(\flat9 \text{ or } \natural9)}$	Minor Seventh Flat Five Bebop Scale	(C,D♭,E♭,F,G♭,G,A♭,B♭)
6. Diminished Scale Choices		
C^{o}	Diminished$_{(W,H)}$ [Octatonic$_{(2,3)}$]	(C,D,E♭,F,G♭,A♭,A,B)

D.1. ($\boldsymbol{D}_{(\emptyset)}$, O)

Group Member	Permutation	Triadic Assignment
i	(1)(2)(3)(4)(5)(6)	$(C^{\Delta})(F^{\Delta})(A^{-})(G^{\Delta})(E^{-})(D^{-})$
a	(1352)	$(C^{\Delta},A^{-},E^{-},F^{\Delta})$
a^2	(15)(23)	$(C^{\Delta},E^{-})(F^{\Delta},A^{-})$
a^{-1}	(1253)	$(C^{\Delta},F^{\Delta},E^{-},A^{-})$
b	(1654)	$(C^{\Delta},D^{-},E^{-},G^{\Delta})$
b^2	(15)(46)	$(C^{\Delta},E^{-})(G^{\Delta},D^{-})$
b^{-1}	(1456)	$(C^{\Delta},G^{\Delta},E^{-},D^{-})$
b^a	(2436)	$(F^{\Delta},G^{\Delta},A^{-},D^{-})$
$(b^a)^2$	(23)(46)	$(F^{\Delta},A-)(G^{\Delta},D^{-})$
$(b^a)^{-1}$	(2634)	$(C^{-},D^{-},A^{-},G^{\Delta})$
ab	(134)(265)	$(C^{\Delta},A^{-},G^{\Delta})(F^{\Delta},D^{-},E^{-})$
$(ab)^{-1}$	(143)(256)	$(C^{\Delta},G^{\Delta},A^{-})(F^{\Delta},E^{-},D^{-})$
ba	(162)(354)	$(C^{\Delta},D^{-},F^{\Delta})(A^{-},E^{-},G^{\Delta})$
$(ba)^{-1}$	(126)(345)	$(C^{\Delta},F^{\Delta},D^{-})(A^{-},G^{\Delta},E^{-})$
$a^{-1}b$	(124)(365)	$(C^{\Delta},F^{\Delta},G^{\Delta})(A^{-},D^{-},E^{-})$
$(a^{-1}b)^{-1}$	(142)(356)	$(C^{\Delta},G^{\Delta},F^{\Delta})(A-,E^{-},D^{-})$
ab^{-1}	(136)(245)	$(C^{\Delta},A^{-},D^{-})(F^{\Delta},G^{\Delta},E^{-})$
$(ab^{-1})^{-1}$	(163)(254)	$(C^{\Delta},D^{-},A-)(F^{\Delta},E^{-},G^{\Delta})$
aba	(15)(26)(34)	$(C^{\Delta},E^{-})(F^{\Delta},D^{-})(A-,G^{\Delta})$
$ab^{-1}a$	(15)(24)(36)	$(C^{\Delta},E^{-})(F^{\Delta},G^{\Delta})(A^{-},D^{-})$
a^2b	(14)(23)(56)	$(C^{\Delta},G^{\Delta})(F^{\Delta},A-)(E^{-},D^{-})$
ab^2	(13)(25)(46)	$(C^{\Delta},A^{-})(F^{\Delta},E^{-})(G^{\Delta},D^{-})$
$a^{-1}b^2$	(12)(35)(46)	$(C^{\Delta},F^{\Delta})(A-,E^{-})(G^{\Delta},D^{-})$
a^2b^3	(16)(23)(45)	$(C^{\Delta},D^{-})(F^{\Delta},A^{-})(G^{\Delta},E^{-})$

D.2. ($\boldsymbol{Hex}_{(3,4)}$, O)

Group Member	Permutation	Triadic Assignment
i	(1)(2)(3)(4)(5)(6)	$(C^{\Delta})(C^{-})(E^{\Delta})(A\flat^{\Delta})(E^{-})(A\flat^{-})$
a	(1352)	$(C^{\Delta},E^{\Delta},E^{-},C^{-})$
a^{2}	(15)(23)	$(C^{\Delta},E^{-})(C^{-},E^{\Delta})$
a^{-1}	(1253)	$(C^{\Delta},C^{-},E^{-},E^{\Delta})$
b	(1654)	$(C^{\Delta},A\flat^{-},E^{-},A\flat^{\Delta})$
b^{2}	(15)(46)	$(C^{\Delta},E^{-})(A\flat^{\Delta},A\flat^{-})$
b^{-1}	(1456)	$(C^{\Delta},A\flat^{\Delta},E^{-},A\flat^{-})$
b^{a}	(2436)	$(C^{-},A\flat^{\Delta},E^{\Delta},A\flat^{-})$
$(b^{a})^{2}$	(23)(46)	$(C^{-},E^{\Delta})(A\flat^{\Delta},A\flat^{-})$
$(b^{a})^{-1}$	(2634)	$(C^{-},A\flat^{-},E^{\Delta},A\flat^{\Delta})$
ab	(134)(265)	$(C^{\Delta},E^{\Delta},A\flat^{\Delta})(C^{-},A\flat^{-},E^{-})$
$(ab)^{-1}$	(143)(256)	$(C^{\Delta},A\flat^{\Delta},E^{\Delta})(C^{-},E^{-},A\flat^{-})$
ba	(162)(354)	$(C^{\Delta},A\flat^{-},C^{-})(E^{\Delta},E^{-},A\flat^{\Delta})$
$(ba)^{-1}$	(126)(345)	$(C^{\Delta},C^{-},A\flat^{-})(E^{\Delta},A\flat^{\Delta},E^{-})$
$a^{-1}b$	(124)(365)	$(C^{\Delta},C^{-},A\flat^{\Delta})(E^{\Delta},A\flat^{-},E^{-})$
$(a^{-1}b)^{-1}$	(142)(356)	$(C^{\Delta},A\flat^{\Delta},C^{-})(E^{\Delta},E^{-},A\flat^{-})$
ab^{-1}	(136)(245)	$(C^{\Delta},E^{\Delta},A\flat^{-})(C^{-},A\flat^{\Delta},E^{-})$
$(ab^{-1})^{-1}$	(163)(254)	$(C^{\Delta},A\flat^{-},E^{\Delta})(C^{-},E^{-},A\flat^{\Delta})$
aba	(15)(26)(34)	$(C^{\Delta},E^{-})(C^{-},A\flat^{-})(E^{\Delta},A\flat^{\Delta})$
$ab^{-1}a$	(15)(24)(36)	$(C^{\Delta},E^{-})(C^{-},A\flat^{\Delta})(E^{\Delta},A\flat^{-})$
$a^{2}b$	(14)(23)(56)	$(C^{\Delta},A\flat^{\Delta})(C^{-},E^{\Delta})(E^{-},A\flat^{-})$
ab^{2}	(13)(25)(46)	$(C^{\Delta},E^{\Delta})(C^{-},E^{-})(A\flat^{\Delta},A\flat^{-})$
$a^{-1}b^{2}$	(12)(35)(46)	$(C^{\Delta},C^{-})(E^{\Delta},E^{-})(A\flat^{\Delta},A\flat^{-})$
$a^{2}b^{3}$	(16)(23)(45)	$(C^{\Delta},A\flat^{-})(C^{-},E^{\Delta})(A\flat^{\Delta},E^{-})$

D.3. Conjugacy classes, ($\boldsymbol{Hex}_{(x,y)}$, O)

Conjugacy Class	Member	Permutation
i	i	(1)(2)(3)(4)(5)(6)
cl(a^2)	a^2	(15)(23)
	b^2	(15)(46)
	$(b^a)^2$	(23)(46)
cl(aba)	aba	(15)(26)(34)
	$ab^{-1}a$	(15)(24)(36)
	a^2b	(14)(23)(56)
	ab^2	(13)(25)(46)
	$a^{-1}b^2$	(12)(35)(46)
	a^2b^3	(16)(23)(45)
cl(ab)	ab	(134)(265)
	$(ab)^{-1}$	(143)(256)
	ba	(162)(354)
	$(ba)^{-1}$	(126)(345)
	$a^{-1}b$	(124)(365)
	$(a^{-1}b)^{-1}$	(142)(356)
	ab^{-1}	(136)(245)
	$(ab^{-1})^{-1}$	(163)(254)
cl(a)	a	(1352)
	a^{-1}	(1253)
	b	(1654)
	b^{-1}	(1456)
	b^a	(2436)
	$(b^a)^{-1}$	(2634)

D.4.($\boldsymbol{Oct}_{(0,1)}$, O)

Group Member	Permutation	Triadic Assignment
i	(1)(2)(3)(4)(5)(6)(7)(8)	$(C^{\Delta})(C^{-})(E\flat^{\Delta})(E\flat^{-})(G\flat^{\Delta})(G\flat^{-})(A^{\Delta})(A^{-})$
g	(1357)(2468)	$(C^{\Delta},E\flat^{\Delta},G\flat^{\Delta},A^{\Delta},)(C^{-},E\flat^{-},G\flat^{-},A^{-})$
g^{2}	(15)(26)(37)(48)	$(C^{\Delta},G\flat^{\Delta})(C^{-},G\flat^{-})(E\flat^{\Delta},A^{\Delta})(E\flat^{-},A^{-})$
g^{-1}	(1753)(2864)	$(C^{\Delta},A^{\Delta},G\flat^{\Delta},E\flat^{\Delta})(C^{-},A^{-},G\flat^{-},E\flat^{-})$
h	(1287)(3465)	$(C^{\Delta},C^{-},A^{-},A^{\Delta})(E\flat^{\Delta},E\flat^{-},G\flat^{-},G\flat^{\Delta})$
h^{2}	(18)(27)(36)(45)	$(C^{\Delta}, A^{-})(C^{-},A^{\Delta})(E\flat^{\Delta}, G\flat^{-})(E\flat^{-},G\flat^{\Delta})$
h^{-1}	(1782)(3564)	$(C^{\Delta}, A^{\Delta},A^{-},C^{-})(E\flat^{\Delta}, G\flat^{\Delta},G\flat^{-},E\flat^{-})$
h^{g}	(1342)(5687)	$(C^{\Delta},E\flat^{\Delta},E\flat^{-},C^{-})(G\flat^{\Delta},G\flat^{-},A^{-},A^{\Delta})$
$(h^{g})^{2}$	(14)(23)(58)(67)	$(C^{\Delta},E\flat^{-})(C^{-},E\flat^{\Delta})(G\flat^{\Delta}, A^{-})(G\flat^{-},A^{\Delta})$
$(h^{g})^{-1}$	(1243)(5786)	$(C^{\Delta},C^{-},E\flat^{-},E\flat^{\Delta})(G\flat^{\Delta} A^{\Delta},A^{-},G\flat^{-})$
gh	(145)(267)	$(C^{\Delta},E\flat^{-},G\flat^{\Delta})(C^{-},G\flat^{-},A^{\Delta})$
$(gh)^{-1}$	(154)(276)	$(C^{\Delta}, G\flat^{\Delta},E\flat^{-})(C^{-},A^{\Delta},G\flat^{-})$
hg	(148)(367)	$(C^{\Delta},E\flat^{-},A^{-})(E\flat^{\Delta},G\flat^{-},A^{\Delta})$
$(hg)^{-1}$	(184)(376)	$(C^{\Delta},A^{-}, E\flat^{-})(E\flat^{\Delta},A^{\Delta},G\flat^{-})$
$g^{-1}h$	(273)(485)	$(C^{-},A^{\Delta},E\flat^{\Delta})(E\flat^{-},A^{-}, G\flat^{\Delta})$
$(g^{-1}h)^{-1}$	(237)(458)	$(C^{-},E\flat^{\Delta},A^{\Delta})(E\flat^{-},G\flat^{\Delta}, A^{-})$
gh^{-1}	(158)(236)	$(C^{\Delta}, G\flat^{\Delta},A^{-})(C^{-},E\flat^{\Delta},G\flat^{-})$
$(gh^{-1})^{-1}$	(185)(263)	$(C^{\Delta}, A^{-},G\flat^{\Delta})(C^{-}, G\flat^{-} ,E\flat^{\Delta})$
ghg	(16)(28)(35)(47)	$(C^{\Delta},G\flat^{-})(C^{-},A^{-})(E\flat^{\Delta},G\flat^{\Delta})(E\flat^{-},A^{\Delta})$
$gh^{-1}g$	(17)(25)(38)(46)	$(C^{\Delta}, A^{\Delta})(C^{-},G\flat^{\Delta})(E\flat^{\Delta},A^{-})(E\flat^{-},G\flat^{-})$
$g^{2}h$	(13)(25)(47)(68)	$(C^{\Delta},E\flat^{\Delta})(C^{-},G\flat^{\Delta})(E\flat^{-},A^{\Delta})(G\flat^{-}, A^{-})$
gh^{2}	(16)(25)(34)(78)	$(C^{\Delta},G\flat^{-})(C^{-},G\flat^{\Delta})(E\flat^{\Delta},E\flat^{-})(A^{\Delta},A^{-})$
$g^{-1}h^{2}$	(12)(38)(47)(56)	$(C^{\Delta}, C^{-})(E\flat^{\Delta}, A^{-})(E\flat^{-},A^{\Delta})(G\flat^{\Delta}, G\flat^{-})$
$g^{2}h^{3}$	(16)(24)(38)(57)	$(C^{\Delta},G\flat^{-})(C^{-},E\flat^{-})(E\flat^{\Delta},A^{-})(G\flat^{\Delta},A^{\Delta})$

D.5. Conjugacy classes, ($\boldsymbol{Oct}_{(x,y)}$, O)

Conjugacy Class	Member	Permutation
()	i	(1)(2)(3)(4)(5)(6)(7)(8)
cl(ghg)	ghg	(16)(28)(35)(47)
	$gh^{-1}g$	(17)(25)(38)(46)
	g^2h	(13)(25)(47)(68)
	gh^2	(16)(25)(34)(78)
	$g^{-1}h^2$	(12)(38)(47)(56)
	g^2h^3	(16)(24)(38)(57)
cl(g^2)	g^2	(15)(26)(37)(48)
	h^2	(18)(27)(36)(45)
	$(h^g)^2$	(14)(23)(58)(67)
cl(gh)	gh	(145)(267)
	$(gh)^{-1}$	(154)(278)
	hg	(148)(367)
	$(hg)^{-1}$	(184)(376)
	$g^{-1}h$	(273)(485)
	$(g^{-1}h)^{-1}$	(237)(458)
	gh^{-1}	(158)(236)
	$(gh^{-1})^{-1}$	(185)(263)
cl(g)	g	(1357)(2468)
	g^{-1}	(1753)(2864)
	h	(1287)(3465)
	h^{-1}	(1782)(3564)
	h^g	(1342)(5687)
	$(h^g)^{-1}$	(1243)(5786)

D.6. *Oh* Reflection, Tetrahedron

i	(1)(2)(3)(4)(5)(6)	$(C^{\Delta})(C^{-})(E^{\Delta})(A\flat^{\Delta})(E^{-})(A\flat^{-})$
m	(15)(23)(46)	$(C^{\Delta},E^{-})(C^{-},E^{\Delta})(A\flat^{\Delta},A\flat^{-})$
$(a)m$	(1253)(46)	$(C^{\Delta},C^{-},E^{-},E^{\Delta})(A\flat^{\Delta},A\flat^{-})$
$(a^{2})m$	(46)	$(A\flat^{\Delta}, A\flat^{-})$
$(a^{-1})m$	(1352)(46)	$(C^{\Delta},E^{\Delta},E^{-},C^{-})(A\flat^{\Delta},A\flat^{-})$
$(b)m$	(1456)(23)	$(C^{\Delta},A\flat^{\Delta},E^{-},A\flat^{-})(C^{-},E^{\Delta})$
$(b^{2})m$	(23)	(C^{-},E^{Δ})
$(b^{-1})m$	(1654)(23)	$(C^{\Delta},A\flat^{-},E^{-},A\flat^{\Delta})(C^{-},E^{\Delta})$
$(b^{a})m$	(15)(2634)	$(C^{\Delta},E^{-})(C^{-},A\flat^{-},E^{\Delta},A\flat^{\Delta})$
$((b^{a})^{2})m$	(15)	(C^{-},E^{Δ})
$((b^{a})^{-1})m$	(15)(2436)	$(C^{\Delta},E^{-})(C^{-},A\flat^{\Delta},E^{\Delta},A\flat^{-})$
$(ab)m$	(124536)	$(C^{\Delta},C^{-},A\flat^{\Delta}, E^{-},E^{\Delta},A\flat^{-})$
$((ab)^{-1})m$	(163542)	$(C^{\Delta},A\flat^{-},E^{\Delta},E^{-},A\flat^{\Delta},C^{-})$
$(ba)m$	(142563)	$(C^{\Delta},A\flat^{\Delta},C^{-},E^{-},A\flat^{-},E^{\Delta})$
$((ba)^{-1})m$	(136524)	$(C^{\Delta},E^{\Delta},A\flat^{-},E^{-},C^{-},A\flat^{\Delta})$
$(a^{-1}b)m$	(134526)	$(C^{\Delta},E^{\Delta},A\flat^{\Delta}, E^{-},C^{-},A\flat^{-})$
$((a^{-1}b)^{-1})m$	(162543)	$(C^{\Delta},A\flat^{-},C^{-},E^{-},A\flat^{\Delta},E^{\Delta})$
$(ab^{-1})m$	(126534)	$(C^{\Delta},C^{-},A\flat^{-},E^{-},E^{\Delta},A\flat^{\Delta})$
$((ab^{-1})^{-1})m$	(1,4,3,5,6,2)	$(C^{\Delta},A\flat^{\Delta},E^{\Delta},E^{-},A\flat^{-},C^{-})$
$(aba)m$	(24)(36)	$(C^{-},A\flat^{\Delta})(E^{\Delta},A\flat^{-})$
$(ab^{-1}a)m$	(26)(34)	$(C^{-},A\flat^{-})(E^{\Delta},A\flat^{\Delta})$
$(a^{2}b)m$	(16)(45)	$(C^{\Delta},A\flat^{-})(A\flat^{\Delta},E^{-})$
$(ab^{2})m$	(12)(35)	$(C^{\Delta},C^{-})(E^{\Delta},E^{-})$
$(a^{-1}b^{2})m$	(13)(25)	$(C^{\Delta},E^{\Delta})(C^{-},E^{-})$
$(a^{2}b^{3})m$	(14)(56)	$(C^{\Delta},A\flat^{\Delta})(E^{-},A\flat^{-})$

D.7. *Oh* Reflection, Cube

i	(1)(2)(3)(4)(5)(6)(7)(8)	$(C^{\Delta})(C^{-})(E\flat^{\Delta})(E\flat^{-})(G\flat^{\Delta})(G\flat^{-})(A^{\Delta})(A^{-})$
k	(12)(34)(56)(78)	$(C^{\Delta},C^{-})(E\flat^{\Delta},E\flat^{-})(G\flat^{\Delta},G\flat^{-})(A^{\Delta},A^{-})$
$(g)k$	(1458)(2367)	$(C^{\Delta},E\flat^{-},G\flat^{\Delta},A^{-})(C^{-},E\flat^{\Delta},G\flat^{-},A^{\Delta})$
$(g^{2})k$	(16)(25)(38)(47)	$(C^{\Delta},G\flat^{-})(C^{-},G\flat^{\Delta})(E\flat^{\Delta},A^{-})(E\flat^{-},A^{\Delta})$
$(g^{-1})k$	(1854)(2763)	$(C^{\Delta},A^{-},G\flat^{\Delta},E\flat^{-})(C^{-},A^{\Delta},G\flat^{-},E\flat^{\Delta})$
$(h)k$	(27)(45)	$(C^{-},A^{\Delta})(E\flat^{-},G\flat^{\Delta})$
$(h^{2})k$	(17)(28)(35)(46)	$(C\Delta,A\Delta)(C^{-},A^{-})(E\flat^{\Delta},G\flat^{\Delta})(E\flat^{-},G\flat^{-})$
$(h^{-1})k$	(18)(36)	$(C^{\Delta},A^{-})(E\flat^{\Delta},G\flat^{-})$
$(h^{g})k$	(14)(67)	$(C^{\Delta},E\flat^{-})(G\flat^{-},A^{\Delta})$
$((h^{g})^{2})k$	(13)(24)(57)(68)	$(C^{\Delta},E\flat^{\Delta})(C^{-},E\flat^{-})(G\flat\Delta,A\Delta)(G\flat-,A-)$
$((h^{g})^{-1})k$	(23)(58)	$(C^{-},E\flat^{\Delta})(A^{-},G\flat^{\Delta})$
$(gh)k$	(134687)(25)	$(C^{\Delta},E\flat^{\Delta},E\flat^{-},G\flat^{-},A^{-},A^{\Delta})(C^{-},G\flat^{\Delta})$
$((gh)^{-1})k$	(16)(287534)	$(C^{\Delta},G\flat^{-})(C^{-},A^{-},A^{\Delta},G\flat^{\Delta},E\flat^{\Delta},E\flat^{-})$
$(hg)k$	(135682)(47)	$(C^{\Delta},E\flat^{\Delta},G\flat^{\Delta},G\flat^{-},A^{-},C^{-})(E\flat^{-},A^{\Delta})$
$((hg)^{-1})k$	(175642)(38)	$(C^{\Delta},A^{\Delta},G\flat^{\Delta},G\flat^{-},E\flat^{-},C^{-})(E\flat^{\Delta},A^{-})$
$(g^{-1}h)k$	(128653)(47)	$(C^{\Delta},C^{-},A^{-},G\flat^{-},G\flat^{\Delta},E\flat^{\Delta})(E\flat^{-},A^{\Delta})$
$((g^{-1}h)^{-1})k$	(124657)(38)	$(C^{\Delta},C^{-},E\flat^{-},G\flat^{-},G\flat^{\Delta},A^{\Delta})(E\flat^{\Delta},A^{-})$
$(gh^{-1})k$	(16)(243578)	$(C^{\Delta},G\flat^{-})(C^{-},E\flat^{-},E\flat^{\Delta},G\flat^{\Delta},A^{\Delta},A^{-})$
$((gh^{-1})^{-1})k$	(178643)(25)	$(C^{\Delta},A^{\Delta},A^{-},G\flat^{-},E\flat^{-},E\flat^{\Delta})(C^{-},G\flat^{\Delta})$
$(ghg)k$	(1548)(2736)	$(C^{\Delta},G\flat^{\Delta},E\flat^{-},A^{-})(C^{-},A^{\Delta},E\flat^{\Delta},G\flat^{-})$
$(gh^{-1}g)k$	(1845)(2637)	$(C^{\Delta},A^{-},E\flat^{-},G\flat^{\Delta})(C^{-},G\flat^{-},E\flat^{\Delta},A^{\Delta})$
$(g^{2}h)k$	(1485)(2673)	$(C^{\Delta},E\flat^{-},A^{-},G\flat^{\Delta})(C^{-},G\flat^{-},A^{\Delta},E\flat^{\Delta})$
$(gh^{2})\mathrm{k}$	(15)(26)	$(C^{\Delta},G\flat^{\Delta})(C^{-},G\flat^{-})$
$(g^{-1}h^{2})k$	(37)(48)	$(E\flat^{\Delta},A^{\Delta})(E\flat^{-},A\flat^{-})$
$(g^{2}h^{3})k$	(1584)(2376)	$(C^{\Delta},G\flat^{\Delta},A^{-},E\flat^{-})(C^{-},E\flat^{\Delta},A^{\Delta},G\flat^{-})$

APPENDIX E. DISCOGRAPHY

COLTRANE, JOHN
"The Father and the Son and the Holy Ghost," 1966. *Meditations*, Impulse! A-9110.

DAMERON, TADD
"Lady Bird," 1949. *The Miles Davis/Tadd Dameron Quintet in Paris Festival international de Jazz*, Columbia JC 34804.

HENDERSON, JOE
"Punjab," 1964. *In 'n Out*, Blue Note BST 84166.

PARKER, CHARLIE
"Ah-Leu-Cha," 1948. *Bird: Master Takes*, Savoy 2201.
"Another Hairdo," 1948. *Bird: Master Takes*, Savoy 2201.
"The Bird," 1956. *Charlie Parker: The Verve Years 1948-1950*, Verve 2501.
"Bird Gets the Worm," 1948. *Bird: Master Takes*, Savoy 2201.
"Bloomdido," 1953. *Charlie Parker: The Verve Years 1948-1950*, Verve 2501/8006/8840.
"Blue Bird," 1948. *Bird: Master Takes*, Savoy 2201.
"Diverse," 1957. *The Charlie Parker Story, no.1*, Verve 8009.
"Kim" no.1, 1956. *The Genius of Charlie Parker, no.3: Now's the Time*, Verve 8005/8840.
"Klaun Stance," 1948. *Bird: Master Takes* Savoy 2201.
"Mohawk," 1956. *Charlie Parker: The Verve Years 1948-1950*, Verve 2501/8006/8840.
"Vista," 1953. *The Charlie Parker Story, no.1*, Verve 8000/8009.
"Warming Up a Riff," 1949. *Bird: Master Takes* Savoy 2201.

MARTINO, PAT
"Joyous Lake," 1976. *Joyous Lake*, Warner Brothers BS 2977.
"Mardi Gras," 1976. *Joyous Lake*, Warner Brothers BS 2977.
"Song Bird," 1976. *Joyous Lake*, Warner Brothers BS 2977.

MONK, THELONIOUS
"Bemsha Swing," co-written by Denzil Best, 1952. *Monk*, Original Jazz Classics OJC-010; Prestige LP 7027.

SCOFIELD, JOHN
"The Beatles," 1979. *Who's Who*, Jive/Novus AN 3018; One Way Records 34512.

SWALLOW, STEVE
"Hotel Vamp," 1974. *Hotel Hello*, ECM 1055.

WHEELER, KENNY
"Ma Belle Hélène," 1990. *The Widow in the Window*, ECM 1417.

APPENDIX F. COPYRIGHT PERMISSION

“The Beatles”

from: John Bishop jbisho8@tigers.lsu.edu
to: sco@johnscofield.com
date: Tue, May 22, 2012 at 7:20 PM
subject: Request for permission to use "The Beatles"
mailed-by: tigers.lsu.edu

Dear Mr. Scofield,
First let me say that I have enjoyed your work for well over twenty-five years. I am currently working on my PhD dissertation in music theory at Louisiana State University. My area of study is mathematics and music. I have worked out an analysis of "The Beatles" that I would like to include in the dissertation and respectfully ask for your permission to provide a lead sheet within the document.

Thank you,

John Bishop

from: John Scofield sco@johnscofield.com
to: "jbisho8@tigers.lsu.edu" <jbisho8@tigers.lsu.edu>
date: Wed, May 23, 2012 at 8:41 AM
subject: Hi John
mailed-by: aol.com
signed-by: mx.aol.com

That's great that you're using the Beatles in your dissertation and thanks for asking about using the lead sheet....permission granted!!!
Thanks for digging the tune!
Good luck!
John Scofield

“The Father, and the Son, and the Holy Ghost” transcription

Confirmation Number: 11005632
Order Date: 06/20/2012
John Bishop
jbisho8@tigers.lsu.edu
+1 (225)7668729

Annual review of jazz studies
- Order detail ID: 62537570
- ISSN: 0731-0641
- Publication year: 2012
- Publication Type: Journal

- Publisher: SCARECROW PRESS, INC.
- Rightsholder: SCARECROW PRESS, INC.
- Author/Editor: David Liebman
- Permission Status: ✔Granted
- Permission type: Republish or display content
- Type of use: Republish in a dissertation
- Republishing organization: Louisiana State University
- Organization status: For-profit
- Republication date: 12/12/2012
- Circulation/ Distribution: 1
- Type of content: Figure/ diagram/ table
- Description of requested content: The Father, the Son and the Holy Ghost
- Page range(s): 170
- Translating to: No Translation
- Requested content's publication date: 12/12/2012

<u>Mick Goodrick reharmonization</u>

From: John Bishop <jbisho8@tigers.lsu.edu>
Date: June 22, 2012 12:07:24 AM EDT
To: <spalm@nbmedia.com>
Subject: Copyright permission
Mr. Palm,
I am writing my dissertation in music theory at Louisiana State University on mathematical permutations of triads and would like to use an example written by Mick Goodrick that was contained in his "Thinking Guitarist" entry to the July 1994 edition of Guitar Player. Whom should I contact to request permission?
Thank you for your help,
John Bishop

On Fri, Jun 22, 2012 at 1:27 PM, Michael Molenda <MMolenda@nbmedia.com> wrote:
Michael Molenda MMolenda@nbmedia.com
Dear John,
Thanks for asking permission.
I have no problem with you using Mick Goodrick's example in your dissertation only, as long as you credit Mick,Guitar Player, and identify the issue (July 1994).
I hope all goes well.
Sincerely,
Mike
Michael Molenda
Editor in Chief, GUITAR PLAYER
Editorial Director, MUSIC PLAYER NETWORK
1111 Bayhill Drive, Suite 125
San Bruno, CA 94066
Office: 650-238-0272
Cell: 415-309-7401

“Hotel Vamp”

from: John Bishop jbisho8@tigers.lsu.edu
to: librarian@wattxtrawatt.com
date: Fri, Nov 2, 2012 at 10:28 PM
subject: Permission to use Hotel Vamp
mailed-by: tigers.lsu.edu

Mr. Swallow,
I am completing my PhD in music theory at Louisiana State University. I am studying mathematics and music. I completed an analysis of 'Hotel Vamp" and respectfully ask permission to include it in my dissertation. The use of your composition will be restricted to the dissertation.
Thank you for your consideration

from: WattXtraWatt wattxtrawatt@wattxtrawatt.com
to: John Bishop <jbisho8@tigers.lsu.edu>
date: Mon, Nov 5, 2012 at 4:13 PM
subject: Re: Permission to use Hotel Vamp

Dear Mr. Bishop,
Sorry for the delay in replying. Steve is touring in Europe but I managed to contact him and he gives you his blessings.
Best Wishes,
The WXW Librarian

“Bemsha Swing”

Thelonious Music Corporation (BMI)
Don Sickler
Second Floor Music (BMI)
130 West 28th Street
NY, NY 10001-6108
p: 212-741-1175
f: 212-627-7611
don@secondfloormusic.com
from: John Bishop jbisho8@tigers.lsu.edu
to: don@secondfloormusic.com
date: Wed, May 23, 2012 at 11:32 PM
subject: Permission to use Bemsha Swing
mailed-by: tigers.lsu.edu

Dear Second Floor Music,
My name is John Bishop and I am currently writing my PhD dissertation in music theory at Louisiana State University. The research uses mathematical group theory to describe a

permutational approach to jazz harmony. I spoke someone in your office about gaining permission to use "Bemsha Swing" as a subject for analysis within the dissertation. She asked that I send a sample of the lead sheet and the analysis. The lead sheet is a transcription I received from a professor at Berklee College in 1987. I completed the Finale image and the analysis. If you have any question regarding the analysis, please let me know.
I respectfully request permission to include "Bemsha Swing" into my dissertation.
Thank you,
John Bishop

2012/8/23 Don Sickler <don@secondfloormusic.com>
Don Sickler
phone 212-741-1175
email don@secondfloormusic.com
Bemsha Swing approved lead sheet
Attachments: DenzilBest-TheloniousMonk_BemshaSwing_Cls.pdf

<u>"Ma Belle Hélène"</u>

from: Brian Shaw bshaw1@lsu.edu
to: Mark Wheeler <whee57h@live.co.uk>
cc: John A Bishop <jbisho8@lsu.edu>
date: Fri, Jun 22, 2012 at 7:55 AM
subject: Hello and question
signed-by: gmail.com

Hello again Mark!
I hope you are doing well. It was so great to see you, Ken, and the rest of the family in London last month. Thanks again for hosting Paula and me.
I have a Doctoral student who is currently writing a very promising dissertation involving some of your Dad's harmonic approaches, and I was hoping he would be able to get permission to use the score to "Ma Belle Hélène in his dissertation.
His name is John Bishop, and I have copied him in on this email. If you would be so kind, please be in touch with him and let him know how Ken feels about this, and if he gives his permission to use the tune in his dissertation.
Thank you very much, and all the best. Looking forward to working on Windmill Tilter with you soon!
Brian

---------- Forwarded message ----------
From: John A Bishop <jbisho8@tigers.lsu.edu>
Date: Thu, Jun 21, 2012 at 11:10 PM

Subject: Wheeler copyright
To: Brian Shaw <bshaw1@lsu.edu>

from:	Mark Wheeler whee57h@live.co.uk
to:	Brian Shaw <bshaw1@lsu.edu>
cc:	jbisho8@tigers.lsu.edu
date:	Tue, Aug 14, 2012 at 10:11 AM
subject:	RE: Hello and question
mailed-by:	live.co.uk

Brian and John

Huge apology to you both.

Please go ahead with the dissertation and let us know if you need anything else.

Good luck John
From: bshaw1@lsu.edu
Date: Mon, 13 Aug 2012 20:01:15 -0500
Subject: Re: Hello and question
To: whee57h@live.co.uk
CC: jbisho8@lsu.edu

BIBLIOGRAPHY

Abercrombie, John. "Modal Management: The Inner Language of Scales." *Guitar Player*. November 1987.

Aebersold, Jamey. *A New Approach to Jazz Improvisation*. New Albany, Indiana: Aebersold Jazz Inc., 1975.

———. *Jazz Handbook*. New Albany: Jamey Aebersold Jazz Inc., 2010.

Aebersold, Jamey and Ken Sloan. *Charlie Parker Omnibook*. New York: Music People, 1978.

Agmon, Eytan. "A Mathematical Model of the Diatonic System." *Journal of Music Theory* 33, no.1 (1989): 1-25.

Albini, Giovanni and Samuele Antonini. "Hamiltonian Cycles in the Topological Dual of the Tonnetz." In *Mathematics and Computation in Music: Second International Conference, MCM 2009, New Haven, CT., USA, June 19-22, 2009, Proceedings*, eds. Elaine Chew, Adrian Childs, and Ching-Hua Chuan. 1-10. Berlin: Springer-Verlag, 2009.

Babbitt, Milton. "Twelve-Tone Invariants as Compositional Determinants." In "Problems of Modern Music. The Princeton Seminar in Advanced Musical Studies." Special issue, *The Musical Quarterly* 46, no. 2 (1960): 246-57.

Baker, David. *Jazz Improvisation*. Chicago: Maher Publications, 1969.

———. *Jazz Improvisation: A Comprehensive Method of Study for all Players*. Van Nuys, CA: Alfred Publishing, 1969.

———. *Advanced Improvisation*. Chicago: Maher Publications, 1974.

Baresel, Alfred. *Jazz-Harmonielehre*. Trossingen: M. Hohner, 1953.

Barr, Stephen. *Experiments in Topology*. New York: Dover Publications, Inc., 1964.

Biamonte, Nicole. "Augmented-Sixth Chords vs. Tritone Substitutes." *Music Theory Online* 14, no.2 (June, 2008). http://mto.societymusictheory.org/issues/mto.08.14.2/mto.08.14.2.biamonte.html (accessed March 1, 2010).

Berliner, Paul F. *Thinking in Jazz:The Infinite Art of Improvisation*. Chicago: University of Chicago Press, 1994.

Block, Steven. "Pitch Class Transformations in Free Jazz." *Music Theory Spectrum* 12, no.2 (1990): 181-202.

Brecker, Michael. *The Michael Brecker Collection*. Transcribed by Carl Coan. Milwaukee: Hal Leonard, n.d.

Bribitzer-Stull, Matthew. "The A♭—C—E Complex: The Origin and function of Chromatic Major Third Collections in Nineteenth-Century Music." *Music Theory Spectrum* 28, no.2 (2006): 167-190.

Budden, F.J. *The Fascination of Groups*. Cambridge: Cambridge University Press, 1972.

———. "Cayley Graphs for Some Well-Known Groups." *The Mathematical Gazette* 69, no. 450 (1985): 271-78.

Campbell, Gary. *Connecting Jazz Theory*. Milwaukee: Hal Leonard, 1998.

———. *Triad Pairs for Jazz*. n.p.: Alfred Publishing, n.d.

Capuzzo, Guy. "Neo-Riemannian Theory and the Analysis of Pop-Rock Music," *Music Theory Spectrum* 26, no.2 (2004): 177-200.

———. "Pat Martino's *The Nature of the Guitar*: An Intersection of Jazz Theory and Neo-Riemannian Theory." *Music Theory Online* 12, no.1 (February, 2006). http://mtosmt.org/issues/mto.06.12.1capuzzo.pdf (accessed March 1, 2010).

Carlton, Larry. "Money Notes" *Guitar Player*, February, 2003.

Carson Berry, David. "The Meaning of "Without": An Exploration of Liszt's Bagatelle ohne Tonart." *19th-Century Music* 29, no.3 (2004): 230-62.

Childs, Adrian P. "Moving Beyond neo-Riemannian Triads: Exploring a Transformational Model for Seventh Chords." *Journal of Music Theory* 42, no.2 (1998): 181-93.

Clough, John, Nora Engebretsen, and Johnathan Kochavi. "Scales, Sets, and Interval Cycles: A Taxonomy." *Music Theory Spectrum* 21, no.1 (1999): 74-104.

Clough, John, Jack Douthett. "Maximally Even Sets." *Journal of Music Theory* 35, nos. 1-2 (1991): 93-173.

Clough, John, Gerald Myerson. "Variety and Multiplicity in Diatonic Systems." *Journal of Music Theory* 29, no.2 (1985): 249-70.

Clough, John. "A Rudimentary Model for Contextual Transposition and Inversion." *Journal of Music Theory* 42, no.2 (1998): 297-306.

Cohn, Richard. "Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions." *Music Analysis* 15, no.1 (1996): 9-40.

———. "Neo-Riemannian Operations, Parsimonious Trichords, and Their *Tonnetz* Representations." *Journal of Music Theory* 41, no.1 (1997): 1-66.

———. "Introduction to Neo-Riemannian Theory: A Survey and a Historical Perspective." *Journal of Music Theory* 42, no.2 (1998): 167-180.

———. “Square Dances with Cubes.” *Journal of Music Theory* 42, no.2 (1998): 283-296.

———. “As Wonderful as Star Clusters: Instruments for Gazing at Tonality in Schubert.” *19th Century Music* 22, no.3 (1999): 213-32.

———. “Weitzmann’s Regions, My Cycles, and Douthett’s Dancing Cubes.” *Music Theory Spectrum* 22, no.1 (2000): 89-103.

———. *Audacious Euphony*. Oxford: Oxford University Press, 2012.

Crans, Alissa S.,Thomas M. Fiore, and Ramon Satyendra. “Musical Actions of Dihedral Groups.” *The American Mathematical Monthly* 116, no.6 (2009): 479-495.

Demsey, David. *John Coltrane Plays Giant Steps*. Milwaukee: Hal Leonard, 1996.

DeVeaux, Scott and Gary Giddins. *Jazz*. New York: W.W. Norton and Company, 1999.

Dixon, John D., and Brian Mortimer. *Permuataion Groups*. Graduate Texts in Mathematics 163. New York: Springer, 1996.

Douthett, Jack, and Peter Steinbach. “Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition.” *Journal of Music Theory* 42, no.2 (1998): 241-63.

Douthett, Jack. “Filtered Point-Symmetry and Dynamical Voice-Leading.” In *Music Theory and Mathematics: Chords, Collections and Transformations*, edited by Jack Douthett, Martha M. Hyde, Charles J. Smith, 72-106. Rochester: University of Rochester Press, 2008.

Feustle, Maristella. “Neo-Riemannian Theory and Post-Bop Jazz: Applications of an Extended Analytical Framework for Seventh Chords.” PhD diss., Bowling Green University, 2005.

Forte, Allen. *The Structure of Atonal Music*. New Haven: Yale University Press, 1973.

Gallian, Joseph. A. *Contemporary Algebra*. 5th ed. Boston: Houghton Mifflin Company, 2002.

Gamer, Carlton and Robin Wilson. “Microtones and Projective Planes.” In *Music and Mathematics: From Pythagoras to Fractals,* edited by. John Fauvel, Raymond Flood and Robin Wilson, 149-61.Oxford: oxford University Press, 2003.

Garzone, George. *The Music of George Garzone and the Triadic Chromatic Approach*. DVD. N.p.: Jody Jazz, n.d.

Gollin, Edward, and Alexander Rehding, eds. *The Oxford Handbook of Neo-Riemannian Music Theories*. Oxford: Oxford University Press, 2011.

Gollin, Edward. “Some Aspects of Three-Dimentional *Tonnetze*.” *Journal of Music Theory* 42, no.2 (1998): 195-206.

Goodrick, Mick. *The Advancing Guitarist*. Milwaukee: Hal Leonard, 1987.

———. “The Thinking Guitarist.” *Guitar Player,* July, 1994.

Harrison, Daniel. *Harmonic Function in Chromatic Music*. Chicago: University of Chicago Press, 1994.

Heuser, Jörg. “Der Gitarrist Pat Martino. Ein Beitrag zur Bedeutung der Gitarre im modernen Jazz.” PhD diss., University of Mainz, 1993.

———. *Pat Martino*: *“Joyous Lake”*. Mainz: BbArking Munckin Music, 1999.

Hook, Julian. “Uniform Triadic Transformations.” *Journal of Music Theory* 46, nos. 1-2 (2002): 57-126.

Hyer, Brian. “Reimag(in)ing Reimann.” *Journal of Music Theory* 39, no.1 (1995): 101-38.

———. “Tonality.” In *The Cambridge History of Western Music Theory*, ed. Thomas Christensen, 726-52. Cambridge: Cambridge University Press, 2002.

Imig, Renate. *Systeme der Funktionsbezeichnung in den Harmonielehren seit Hugo Riemann*. Düsseldorf: Gesellschaft zur Förderung der systematischen Musikwissenschaft, 1970.

Julien, Patricia A. “The Structural Function of Harmonic Relations in Wayne Shorter’s Early Compositions: 1953-1963.” PhD. Diss., University of Maryland, 2003.

Keller, Gary. *The Jazz Chord/Scale Handbook*. Rottenburg: Advance Music, 2002.

King, Martin Luther, Jr. “Opening Address to the 1964 Berlin Jazz Festival.” WPFW News, Washington D.C, August 23, 2002.

Kopp, David. *Chromatic Transformations in Nineteenth-Century Music*. Cambridge Studies in Music Theory and Analysis. Edited by Ian Bent. Cambridge: Cambridge University Press, 2002.

Lambert, Philip. “Isographies and Some Klumpenhouwer Networks they Involve.” *Music Theory Spectrum* 24, no.2 (2002): 165-95.

Levine, Mark. *The Jazz Theory Book*. Petaluma, CA.: Sher Music Company, 1995.

Lewin, David. “Klumpenhouwer Networks and Some Isographies that Involve Them.” *Music Theory Spectrum* 12, no.1 (1990): 83-120.

———. “Some Compositional Uses of Projective Geometry.” *Perspectives of New Music* 42, no.2 (2004): 12-63.

———. *Generalized Musical Intervals and Transformations*. Oxford: Oxford University Press, 2007.

Liebman, David, Richard Beirach, Frank Tusa, Jeff Williams, and Badal Roy. *Lookout Farm: a Case Study of Improvisation for Small Jazz Group*. N.p.: Alamo Publications, 1977.

———. “John Coltrane’s *Meditations* Suite: A Study in Symmetry.” *Annual Review of Jazz Studies* 8 (1996): 167-180.

———. *A Chromatic Approach to Jazz Harmony and Melody*. Rottenburg: Advance Music, 2001.

Lippius, Johannes. *Synopsis of New Music (Synopsis Musicae Novae)*. Strassberg: 1612. Translated by Benito V. Rivera (Colorado Springs: Colorado College Music Press, 1977).

Martin, Henry. “Jazz Harmony: A Syntactic Background.” *Annual Review of Jazz Studies* 4 (1986): 9-31.

———. “Jazz Theory: An Overview.” *Annual Review of Jazz Studies* 8 (1996): 1-17.

Martino, Pat. *Quantum Guitar: Analysis of a Tune*. Miami: Warner Brothers Publications Inc., 1996.

———. “Nature of the Guitar.” http://www.patmartino.net/nature_of_guitar_Mac_061116.html (accessed March 9, 2010).

Masakowski, Steve. “Major Symmetry, Diminished Treasure” *Guitar Player*, July, 1996.

Mazzola, Guerino. *The Topos of Music: Geometeric Logic of Concepts, Theory, and Performance*. Basel: Birkhäuser, 2002.

Mazzola, Guerino and Moreno Andreatta. “Diagrams, Gestures and Formulae in Music,” *Journal of Mathematics and Music* 1, no.1 (2007): 21-46.

Mauleón-Santana, Rebeca. *101 Montunos*. Petaluma, CA.: Sher Music Company, 1999.

Mendelson, Bert. *Introduction to Topology*. 3rd ed. New York: Dover Publications Inc., 1990.

Messiaen, Olivier. *La technique de mon langage musical*. Translated by John Satterfeld (Paris: Alphonse Leduc, 1956).

Mickelsen, William C, trans. and ed. *Hugo Riemann’s Theory of Harmony: A Study by William C. Mickelsen and History of Music Theory, Book III by Hugo Riemann*. Lincoln: University of Nebraska Press, 1977.

Miller, Ron. *Modal Jazz: Composition and Harmony*. vol 1. Rottenburg: Advance Music, 1996.

Moore, John T. *Elements of Abstract Algebra*. 2nd ed. New York: Macmillan Company, 1967.

Naus, Wayne. *Beyond Functional Harmony*. n.p.: Advance Music, 1998.

Nettles, Barrie, Richard Graf. *The Chord Scale Theory and Jazz Harmony*. Rottenberg: Advance Music, 1997.

O’Donnell, Shaugn. “Klumpenhouwer Networks, Isography, and the Molecular Metaphor.” *Intégral* 12 (1998): 53-79.

Peck, Robert. “Toward an Interpretation of Xenakis’s *Nomos Alpha*.” *Perspectives of New Music* 41, no.1 (2003): 66-118.

———. “Aspects of Recursion in M-Inclusive Networks.” *Intégral*, nos. 18-19 (2004): 25-70.

———. “A Response to Jack Douthett’s ‘Filtered Point-Symmetry and Dynamical Voice-Leading.’” Presented at The Conference on Musical Systems in Memory of John Clough, University of Chicago, Chicago, IL, July 8, 2005.

———. “Generalized Commuting Groups.” Journal of Music Theory 54, no.2 (2010): 143-77.

———. “Imaginary Transformations.” *Journal of Mathematics and Music* 4, no. 3 (2010): 157-71.

———. “A GAP Tutorial for Transformational Music Theory.” *Music Theory Online* 17, no.1 (April, 2011). http://www.mtosmt.org/issues/mto.11.17.1/mto.11.17.1.peck.html (accessed September 1, 2012).

Pressing, Jeff. “Towards an Understanding of Scales in Jazz.” *Jazz Research*, no. 9 (1978): 25-35.

Rahn, John. *Basic Atonal Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1981.

———. “Cool Tools: Polysemic and Non-Commutative Nets, Subchain Decompositions and Cross-Projecting Pre-Orders, Object-Graphs, Chain-Hom-Sets and Chain-Label-Hom-Sets, Forgetful Functors, free Categories of a Net, and Ghosts.” *Journal of Mathematics and Music* 1, no.1 (2007): 7-22.

Russom, Phillip Wade. “A Theory of Pitch Organization for the Early Works of Maurice Ravel.” PhD. dissertation, Yale University, 1985.

Russell, George. *The Lydian Chromatic Concept of Tonal Organization for Improvisation*. New York: Concept Publishing Company, 1959.

Santa, Matthew. “Nonatonic Progressions in the Music of John Coltrane.” *Annual Review of jazz Studies* 13 (2003): 13-25.

Saslaw, Janna K. “Gottfried Weber’s Cognitive Theory of Harmonic Progression.” *Studies in Music from the University of Western Ontario* 13 (1991): 121-44.

———. “Gottfried Weber and Multiple Meaning.” *Theoria* 5 (1990-1): 74-103.

———. “Gottfried Weber and the Concept of *Mehrdeutigkeit*.” PhD. diss., Columbia University, 1992.

Schuster-Craig, John. “An Eighth Mode of Limited Transposition.” *The Music Review* 51, no. 4 (1990): 296-306.

Scott, W.R. *Group Theory*. Englewood Cliffs, NJ: Prentice-Hall, Inc., 1964.

Shaperio, Nat, and Nat Hentoff. *Hear Me Talkin' To Ya*. Mineola, NY: Dover, 1955.

Sifter, Suzanna. *Using Upper-Structure Triads*. Boston: Berklee Press, 2011.

Stewart, B.M. *Adventures Among the Toroids: A Study of Orientable Polyhedra with Regular Faces*. 2nd ed. Okemos, MI.: B.M. Stewart, 1980.

Stoecker, Philip."Klumpenhouwer Networks, Trichords, and Axial Isography." *Music Theory Spectrum* 24, no.2 (2002): 231-45.

Straus, Joseph N. *Introduction to Post-Tonal Theory*. 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 2004.

Strunk, Steven. "Wayne Shorter's *Yes and No*: An Analysis." *Tijdschrift voor Muziektheorie* 8, no.1 (2003): 40-56.

———. "Notes on Harmony in Wayne Shorter's Composition, 1964-67." *Journal of Music Theory* 49, no.2 (2005): 301-332.

Soderberg, Stephen. "Z-Related Sets as Dual Inversions." *Journal of Music Theory* 39, no.1 (1995): 77-100.

Tymoczko, Dmitri. "The Consecutive Semitone Restraint: A Link Between Impressionism and Jazz." *Intégral*, no.11 (1997): 135-179.

———. "Scale Networks and Debussy." *Journal of Music Theory* 48, no.2 (2004): 219-294.

———. *A Geometry of Music*. Oxford: Oxford University Press, 2011.

———. "The Generalized *Tonnetz*." *Journal of Music Theory* 56, no.1 (2012): 1-52.

Waters, Keith J. and J. Kent Williams. "Modes, Scales, Functional Harmony and Non-Functional Harmony in the Compositions of Herbie Hancock." *Journal of Music Theory* 49, no.2 (2005): 333-357.

———. "Modeling Diatonic, Acoustic, Hexatonic, and Octatonic Harmonies and Progressions in 2- and 3-Dimensional Pitch Spaces; or Jazz Harmony after 1960."*Music Theory Online* 16, no.3 (May, 2010). http://mto.societyofmusictheoty.org/mto.10.16.3.waters.williams.html (accessed June 2, 2010).

Weber, Gottfried. *Versuch einer geordneten Theorie der Tonsetzkunst*. 3rd ed. Mainz: B. Schott, 1830-2.

Weiskopf, Walt and Ramon Ricker. *Giant Steps: A Player's Guide to Coltrane's Harmony for all Instrumentalists*. New Albany, IN: Jamey Aebersold Jazz Inc., 1991.

———. *The Augmented Scale in Jazz*. New Albany, IN: Jamey Aebersold Jazz Inc., 1993.

Weiskopf, Walt. *Intervallic Improvisation, the Modern Sound: A Step Beyond Linear Improvisation*. New Albany, IN: Jamey Aerbersold Jazz Inc., 1995.

Weitzmann, Carl Friedrich. *Der übermässige Dreiklang*. Berlin: T. Trautweinschen, 1853.

Xenakis, Iannis. *Formalized Music*. Revised ed. Stuyvesant, New York: Pendragon, 1992.

Yamaguchi, Masaya. *Symmetrical Scales for Jazz Improvisation*. Revised ed. New York: Masaya Music Services, 2006.

———. "A Creative Approach to Multi-Tonic Changes: Beyond Coltrane's Harmonic Formula." *Annual Review of Jazz Studies* 12 (2002): 147-67.

Zweifel, Paul F. "Generalized Diatonic and Pentatonic Scales: A Group-Theoretic Approach." *Perspectives of New Music* 34, no.1 (1996):140-161.

VITA

John Bishop received the Bachelor of Music, cum laude, in guitar performance from Berklee College of music in 1990, the Master of Music in jazz studies from University of Louisville in 2004, and the Doctor of Philosophy in music theory from Louisiana State University in 2012. As a guitarist, he enjoyed an active career in the San Francisco Bay Area. He has performed with jazz artists Steve Erquiaga, Jimmy Raney, and David Liebman and gospel artists Kim Burrell, Bobby Jones, and Lonnie Hunter. He taught at the Jamey Aebersold Jazz Workshops; jazz guitar ensemble at the University of Louisville; and honors Aural Skills, Rudiments of Music, Improvisation, small-group Jazz Ensembles, and Jazz Guitar at Louisiana State University.